

Representations of Yangians with Gelfand-Zetlin bases

By *Maxim Nazarov* at York and *Vitaly Tarasov* at St. Petersburg

We study a certain family of finite-dimensional modules over the Yangian $Y(\mathfrak{gl}_N)$. The Yangian $Y(\mathfrak{gl}_N)$ is a canonical deformation of the universal enveloping algebra $U(\mathfrak{gl}_N[u])$ in the class of Hopf algebras [D1]. The classification of the irreducible finite-dimensional $Y(\mathfrak{gl}_N)$ -modules has been obtained in [D2] by generalizing the results of the second author [T1,T2]. These modules are parametrized by all pairs formed by a sequence of monic polynomials $P_1(u), \dots, P_{N-1}(u) \in \mathbb{C}[u]$ and a power series $f(u) \in 1 + u^{-1} \mathbb{C}[[u^{-1}]]$.

The algebra $Y(\mathfrak{gl}_N)$ comes equipped with a distinguished maximal commutative subalgebra $A(\mathfrak{gl}_N)$. This subalgebra is generated by the centres of all algebras in the chain

$$Y(\mathfrak{gl}_1) \subset Y(\mathfrak{gl}_2) \subset \dots \subset Y(\mathfrak{gl}_N).$$

Continuing [C], we study finite-dimensional $Y(\mathfrak{gl}_N)$ -modules with a semisimple action of the subalgebra $A(\mathfrak{gl}_N)$. We will call these modules tame. We provide a characterization of the irreducible tame modules in terms of the polynomials $P_1(u), \dots, P_{N-1}(u)$; see Theorem 4.1.

It turns out that the spectrum of the action of the subalgebra $A(\mathfrak{gl}_N)$ in every irreducible tame module is simple. The eigenbases of $A(\mathfrak{gl}_N)$ in the latter modules are called Gelfand-Zetlin bases. In Section 3 we provide explicit formulas for the action of the generators of the algebra $Y(\mathfrak{gl}_N)$ introduced in [D2] on the vectors of these bases. Moreover, each of these bases contains the vector ξ_0 called singular. For every vector ξ of the Gelfand-Zetlin basis we point out an element $b \in Y(\mathfrak{gl}_N)$ such that $b \cdot \xi_0 = \xi$.

The algebra $Y(\mathfrak{gl}_N)$ admits a homomorphism onto the universal enveloping algebra $U(\mathfrak{gl}_N)$. Denote this homomorphism by π_N . The image of the centre of $Y(\mathfrak{gl}_N)$ with respect to π_N coincides with the centre $Z(\mathfrak{gl}_N)$ of the algebra $U(\mathfrak{gl}_N)$. So any finite-dimensional $Y(\mathfrak{gl}_N)$ -module obtained via the homomorphism π_N is tame. The results of Section 3 can be regarded as a generalization of the classical formulas of [GZ]. For more details on the connection between the formulas of [GZ] and the Yangian $Y(\mathfrak{gl}_N)$ see our publication [NT]. It has inspired the publication [M].

For each $M = 0, 1, 2, \dots$ there is a homomorphism of $Y(\mathfrak{gl}_N)$ to the commutant in $U(\mathfrak{gl}_{M+N})$ of the subalgebra $U(\mathfrak{gl}_M)$. The image of this homomorphism along with the centre $Z(\mathfrak{gl}_M)$ generates the commutant [O]. For any dominant integral weights λ and μ of the Lie algebras \mathfrak{gl}_{M+N} and \mathfrak{gl}_M denote by $V_{\lambda,\mu}$ the subspace in the irreducible \mathfrak{gl}_{M+N} -module of the highest weight λ formed by all singular vectors with respect to \mathfrak{gl}_M of the weight μ . The latter homomorphism makes $V_{\lambda,\mu}$ into an irreducible tame $Y(\mathfrak{gl}_N)$ -module.

Furthermore, for any $h \in \mathbb{C}$ there is an automorphism of the algebra $Y(\mathfrak{gl}_N)$ preserving the subalgebra $A(\mathfrak{gl}_N)$. Applying this automorphism to an irreducible finite-dimensional $Y(\mathfrak{gl}_N)$ -module V amounts to changing u by $u + h$ in the

respective polynomials $P_1(u), \dots, P_{N-1}(u)$ and the series $f(u)$. Denote the resulting $Y(\mathfrak{gl}_N)$ -module by $V(h)$. We prove that up to the choice of the series $f(u)$ every irreducible tame $Y(\mathfrak{gl}_N)$ -module splits into a tensor product of modules of the form $V_{\lambda, \mu}(h)$; see Theorem 4.1.

Consider the $Y(\mathfrak{gl}_N)$ -module $V(h)$ where the module V is obtained from the fundamental \mathfrak{gl}_N -module \mathbb{C}^N via the homomorphism π_N . The above stated splitting property of any irreducible tame $Y(\mathfrak{gl}_N)$ -module was established in [C], Theorem 3.9 under two extra conditions. First, the spectrum of the action of the algebra $A(\mathfrak{gl}_N)$ in this module was assumed to be simple. Second, this module was assumed to be a subquotient of the module $V(h) \otimes \dots \otimes V(h')$ where the number of tensor factors is less than N . It was also conjectured in [C] that both these conditions could be removed. So we confirm this conjecture in the present article.

The Yangian $Y(\mathfrak{gl}_N)$ can be viewed [D3] as a degeneration of the quantum universal enveloping algebra $U_q(\widehat{\mathfrak{gl}_N})$. All the results presented in this article have their quantum counterparts. We will give them in a forthcoming publication. Note that Theorem 4.11 from [GRV] is the counterpart of our formulas for the action of $Y(\mathfrak{gl}_N)$ in the module $V(h) \otimes \dots \otimes V(h')$ where $V = \mathbb{C}^N$ as above while the parameters $h, \dots, h' \in \mathbb{C}$ are in general position; see Theorem 2.9 and Corollary 3.9 here.

It is our pleasure to thank Ivan Cherednik and Grigori Olshanski for numerous creative conversations. The first author was supported by the University of Wales Research Fellowship. The second author should like to thank the Mathematics Department of the University of Wales at Swansea for hospitality.

1. Preliminaries on Yangians

In this section we gather several known facts about the *Yangian* of the Lie algebra \mathfrak{gl}_N . This is a complex associative unital algebra $Y(\mathfrak{gl}_N)$ with the countable set of generators $T_{ij}^{(s)}$ where $s = 1, 2, \dots$ and $i, j = 1, \dots, N$. The defining relations in the algebra $Y(\mathfrak{gl}_N)$ are

$$(1.1) \quad [T_{ij}^{(r)}, T_{kl}^{(s+1)}] - [T_{ij}^{(r+1)}, T_{kl}^{(s)}] = T_{kj}^{(r)} T_{il}^{(s)} - T_{kj}^{(s)} T_{il}^{(r)}; \quad r, s = 0, 1, 2, \dots$$

where $T_{ij}^{(0)} = \delta_{ij} \cdot 1$. We will also use the following matrix form of these relations.

Let $E_{ij} \in \text{End}(\mathbb{C}^N)$ be the standard matrix units. Introduce two formal variables u, v and consider the *Yang R-matrix*

$$R(u, v) = (u - v) \cdot \text{id} + \sum_{i, j=1}^N E_{ij} \otimes E_{ji} \in \text{End}(\mathbb{C}^N)^{\otimes 2} [u, v].$$

Introduce the formal power series in u^{-1}

$$T_{ij}(u) = T_{ij}^{(0)} + T_{ij}^{(1)} u^{-1} + T_{ij}^{(2)} u^{-2} + \dots$$

and combine all these series into the single element

$$T(u) = \sum_{i, j=1}^N E_{ij} \otimes T_{ij}(u) \in \text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N) [[u^{-1}]].$$

For any associative unital algebra X denote by ι_s its embedding into a finite tensor product $X^{\otimes n}$ as the s -th tensor factor:

$$\iota_s(x) = 1^{\otimes (s-1)} \otimes x \otimes 1^{\otimes (n-s)}, \quad x \in X; \quad s = 1, \dots, n.$$

Introduce also the formal power series with the coefficients in $\text{End}(\mathbb{C}^N)^{\otimes 2} \otimes Y(\mathfrak{gl}_N)$

$$T_1(u) = \iota_1 \otimes \text{id}(T(u)), \quad T_2(v) = \iota_2 \otimes \text{id}(T(v)), \quad R_{12}(u, v) = R(u, v) \otimes 1.$$

Then the relations (1.1) can be rewritten as

$$(1.2) \quad R_{12}(u, v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u, v)$$

The relations (1.2) imply that for any formal series $f(u) \in 1 + u^{-1} \mathbb{C}[[u^{-1}]]$ the assignment of the generating series $T_{ij}(u) \mapsto f(u) \cdot T_{ij}(u)$ determines an automorphism of the algebra $Y(\mathfrak{gl}_N)$. We will denote by ω_f this automorphism. The element $T(u)$ of the algebra $\text{End}(\mathbb{C}^N) \otimes Y(\mathfrak{gl}_N)[[u^{-1}]]$ is invertible; denote

$$(1.3) \quad T(u)^{-1} = \tilde{T}(u) = \sum_{i,j=1}^N E_{ij} \otimes \tilde{T}_{ij}(u).$$

Then the relations (1.2) along with the equality $R(u, v) R(-u, -v) = 1 - (u - v)^2$ imply that the assignment $T_{ij}(u) \mapsto \tilde{T}_{ij}(-u)$ determines an automorphism of the algebra $Y(\mathfrak{gl}_N)$. We will denote by σ_N this automorphism; it is clearly involutive. We will also make use of the automorphism τ_N determined by the assignment

$$T_{ij}(u) \mapsto \tilde{T}_{N-j+1, N-i+1}(u).$$

Lemma 1.1. *Suppose that $i \neq l$ and $k \neq j$. Then the coefficients of the series $T_{ij}(u)$ commute with those of $\tilde{T}_{kl}(u)$.*

Proof. Multiplying (1.2) on the left and on the right by $T_2(v)^{-1}$ we obtain the equality

$$T_1(u) R_{12}(u, v) T_2(v)^{-1} = T_2(v)^{-1} R_{12}(u, v) T_1(u).$$

Since $i \neq l$ and $k \neq j$ by multiplying the latter equality by $E_{ii} \otimes E_{kk} \otimes 1$ on the left and by $E_{jj} \otimes E_{ll} \otimes 1$ on the right we obtain that

$$E_{ij} \otimes E_{kl} \otimes [T_{ij}(u), \tilde{T}_{kl}(-v)] = 0 \quad \square$$

Due to (1.1) every sequence of pairwise distinct indices $\mathbf{k} = (k_1, \dots, k_M)$ where $1 \leq k_i \leq M + N$, determines an embedding of algebras

$$(1.4) \quad \varphi_{\mathbf{k}} : Y(\mathfrak{gl}_M) \rightarrow Y(\mathfrak{gl}_{M+N}) : T_{ij}(u) \mapsto T_{k_i k_j}(u).$$

We will also make use of the embedding of the same algebras $\psi_{\mathbf{k}} = \sigma_{M+N} \varphi_{\mathbf{k}} \sigma_M$. The embedding $\varphi_{\mathbf{k}}$ with $\mathbf{k} = (1, \dots, M)$ will be called *standard*. We will identify the algebra $Y(\mathfrak{gl}_M)$ and its image in $Y(\mathfrak{gl}_{M+N})$ with respect to the standard embedding.

Now let $\mathbf{k} = (k_1, \dots, k_M)$ and $\mathbf{l} = (l_1, \dots, l_N)$ be two sequences such that

$$(1.5) \quad \{k_1, \dots, k_M\} \sqcup \{l_1, \dots, l_N\} = \{1, \dots, M + N\}.$$

Corollary 1.2. *The images of the embeddings $\varphi_{\mathbf{k}}$ and ψ_1 in $Y(\mathfrak{gl}_{M+N})$ commute.*

Proof. The image of the mapping ψ_1 coincides with that of $\sigma_{M+N} \varphi_1$ and we have $\sigma_{M+N} \varphi_1(T_{mn}(u)) = \tilde{T}_{l_m l_n}(u)$. Due to Lemma 1.1 the coefficients of the series $T_{k_i k_j}(u)$ commute with those of $\tilde{T}_{l_m l_n}(u)$ for all possible i, j, m, n \square

Consider the elements E_{ij} as generators of the Lie algebra \mathfrak{gl}_N . The algebra $Y(\mathfrak{gl}_N)$ contains the universal enveloping algebra $U(\mathfrak{gl}_N)$ as a subalgebra: due to (1.1) the assignment $E_{ji} \mapsto T_{ij}^{(1)}$ defines the embedding. Moreover, there is a homomorphism

$$(1.6) \quad \pi_N : Y(\mathfrak{gl}_N) \rightarrow U(\mathfrak{gl}_N) : T_{ij}(u) \mapsto \delta_{ij} + E_{ji} u^{-1}.$$

Note that this homomorphism is by definition identical on the subalgebra $U(\mathfrak{gl}_N)$. By (1.3) the automorphism σ_N of the algebra $Y(\mathfrak{gl}_N)$ is also identical on $U(\mathfrak{gl}_N)$.

Let $Z(\mathfrak{gl}_M)$ denote the center of the universal enveloping algebra $U(\mathfrak{gl}_M)$. By the definition (1.4) the image $\varphi_{\mathbf{k}}(U(\mathfrak{gl}_M))$ is contained in $U(\mathfrak{gl}_{M+N})$. For the proof of the following proposition see [O].

Proposition 1.3. *The centralizer of the subalgebra $\varphi_{\mathbf{k}}(U(\mathfrak{gl}_M))$ in $U(\mathfrak{gl}_{M+N})$ coincides with*

$$\varphi_{\mathbf{k}}(Z(\mathfrak{gl}_M)) \cdot \pi_{M+N} \psi_1(Y(\mathfrak{gl}_N)).$$

We will now describe an alternative set of generators for the algebra $Y(\mathfrak{gl}_N)$. These generators were introduced in [D2], and we will call them Drinfeld generators. Let $\mathbf{i} = (i_1, \dots, i_m)$ and $\mathbf{j} = (j_1, \dots, j_m)$ be two sequences of indices such that

$$(1.7) \quad 1 \leq i_1 < \dots < i_m \leq N \quad \text{and} \quad 1 \leq j_1 < \dots < j_m \leq N.$$

Consider the sum over all permutations g of $1, 2, \dots, m$

$$Q_{\mathbf{ij}}(u) = \sum_g T_{i_1 j_{g(1)}}(u) T_{i_2 j_{g(2)}}(u-1) \dots T_{i_m j_{g(m)}}(u-m+1) \cdot \text{sign } g;$$

here the series in $(u-1)^{-1}, \dots, (u-m+1)^{-1}$ should be re-expanded in u^{-1} . Denote

$$(1.6) \quad A_m(u) = Q_{\mathbf{ii}}(u) \quad \text{for } \mathbf{i} = (1, \dots, m).$$

We will set $A_0(u) = 1$. The series $A_N(u)$ is called the *quantum determinant* for the algebra $Y(\mathfrak{gl}_N)$. The following proposition is well known; a detailed proof can be found in [MNO], Section 2.

Proposition 1.4. *The coefficients at u^{-1}, u^{-2}, \dots of the series $A_N(u)$ are free generators for the centre of the algebra $Y(\mathfrak{gl}_N)$.*

Consider the ascending chain of algebras

$$(1.8) \quad Y(\mathfrak{gl}_1) \subset Y(\mathfrak{gl}_2) \subset \dots \subset Y(\mathfrak{gl}_N)$$

determined by the standard embeddings. Denote by $A(\mathfrak{gl}_N)$ the subalgebra of $Y(\mathfrak{gl}_N)$ generated by the centres of all the algebras in (1.8); this subalgebra is

commutative. By Proposition 1.4 the coefficients of the series $A_1(u), \dots, A_N(u)$ generate the subalgebra $A(\mathfrak{gl}_N)$. The subalgebra $A(\mathfrak{gl}_N)$ is maximal commutative in $Y(\mathfrak{gl}_N)$. This fact is contained in [C], Theorem 2.2. However, we will not use this fact in the present article. Now for every $m = 1, \dots, N-1$ denote

$$(1.9) \quad B_m(u) = Q_{\mathbf{i}\mathbf{j}}(u), \quad C_m(u) = Q_{\mathbf{j}\mathbf{i}}(u), \quad D_m(u) = Q_{\mathbf{j}\mathbf{j}}(u)$$

where $\mathbf{i} = (1, \dots, m)$ and $\mathbf{j} = (1, \dots, m-1, m+1)$. The coefficients of the series

$$A_m(u), \quad B_m(u), \quad C_m(u); \quad m = 1, \dots, N-1$$

along with those of $A_N(u)$ generate the algebra $Y(\mathfrak{gl}_N)$; see [D2], Example. We will call the coefficients of all these series the *Drinfeld generators* for $Y(\mathfrak{gl}_N)$.

Remark. Consider the fixed point subalgebra in $Y(\mathfrak{gl}_N)$ with respect to all automorphisms of the form ω_f . This algebra is called the *Yangian* for the simple Lie algebra \mathfrak{sl}_N and denoted by $Y(\mathfrak{sl}_N)$. The coefficients of all the series

$$A_{m-1}(u-1) A_m^{-1}(u-1) A_m^{-1}(u) A_{m+1}(u), \quad A_m^{-1}(u) B_m(u), \quad C_m(u) A_m^{-1}(u)$$

with $m = 1, \dots, N-1$ are generators of the algebra $Y(\mathfrak{sl}_N)$; see again [D2], Example. The algebra $Y(\mathfrak{gl}_N)$ is isomorphic to the tensor product of its centre and the algebra $Y(\mathfrak{sl}_N)$; see [MNO], Section 2 for the proof of this claim.

Now let \mathbf{i} and \mathbf{j} be any two sequences of indices satisfying the condition (1.7). Introduce the increasing sequences $\check{\mathbf{i}} = (i_{m+1}, \dots, i_N)$ and $\check{\mathbf{j}} = (j_{m+1}, \dots, j_N)$ such that

$$\{i_1, \dots, i_m\} \sqcup \{i_{m+1}, \dots, i_N\} = \{j_1, \dots, j_m\} \sqcup \{j_{m+1}, \dots, j_N\} = \{1, \dots, N\}.$$

Let ε be the sign of the permutation $(i_1, \dots, i_n) \mapsto (j_1, \dots, j_n)$. Introduce also the sequences

$$\begin{aligned} \tilde{\mathbf{i}} &= (N - i_N + 1, \dots, N - i_{m+1} + 1), \\ \tilde{\mathbf{j}} &= (N - j_N + 1, \dots, N - j_{m+1} + 1). \end{aligned}$$

Lemma 1.5. *In the algebra $Y(\mathfrak{gl}_N)[[u^{-1}]]$ we have the equalities*

$$(1.10) \quad \sigma_N(Q_{\mathbf{i}\mathbf{j}}(u)) = \varepsilon \cdot Q_{\tilde{\mathbf{j}}\tilde{\mathbf{i}}}(-1-u) / A_N(m-1-u),$$

$$(1.11) \quad \tau_N(Q_{\mathbf{i}\mathbf{j}}(u)) = \varepsilon \cdot Q_{\tilde{\mathbf{i}}\tilde{\mathbf{j}}}(u-m) / A_N(u).$$

Proof. Let $H \in \text{End}(\mathbb{C}^N)^{\otimes m}$ denote the antisymmetrization map normalized so that $H^2 = m! H$. Then introduce the map

$$H_m = \iota_1 \dots \iota_m(H) \in \text{End}(\mathbb{C}^N)^{\otimes m}.$$

Introduce also the formal power series with the coefficients in $\text{End}(\mathbb{C}^N)^{\otimes N} \otimes Y(\mathfrak{gl}_N)$

$$T_1(u) = \iota_1 \otimes \text{id}(T(u)), \dots, T_N(v) = \iota_N \otimes \text{id}(T(u)).$$

Then we have

$$(1.12) \quad T_1(u) T_2(u-1) \dots T_m(u-m+1) \cdot (H_m \otimes 1) = \\ (H_m \otimes 1) \cdot T_m(u-m+1) \dots T_2(u-1) T_1(u);$$

see [MNO], Section 2 for the proof of this equality. In particular, when $m = N$ this equality implies that

$$T_1(u) T_2(u-1) \dots T_N(u-N+1) \cdot (H_N \otimes 1) = H_N \otimes A_N(u).$$

Therefore we have

$$(1.13) \quad T_{m+1}(u-m) T_{m+2}(u-m-1) \dots T_N(u-N+1) \cdot (H_N \otimes 1) = \\ \tilde{T}_m(u-m+1) \dots \tilde{T}_2(u-1) \tilde{T}_1(u) \cdot (H_N \otimes A_N(u)).$$

By multiplying the latter equality on the left by $I_N \otimes 1$ where I_N equals

$$E_{i_m i_m} \otimes \dots \otimes E_{i_1 i_1} \otimes E_{j_{m+1} j_{m+1}} \otimes \dots \otimes E_{j_N j_N}$$

we obtain that

$$\varepsilon \cdot Q_{\mathbf{j}\mathbf{i}}(u-m) = \sigma_N(Q_{\mathbf{i}\mathbf{j}}(m-1-u)) A_N(u).$$

Replacing here the variable u by $m-1-u$ we obtain (1.10). The equality (1.12) also implies

$$\tilde{T}_m(u-m+1) \dots \tilde{T}_2(u-1) \tilde{T}_1(u) \cdot (H_m \otimes 1) = \\ (H_m \otimes 1) \cdot \tilde{T}_1(u) \tilde{T}_2(u-1) \dots \tilde{T}_m(u-m+1).$$

Therefore by multiplying the equality (1.13) on the left by $J_N \otimes 1$ where J_N equals

$$E_{N-j_1+1, N-j_1+1} \otimes \dots \otimes E_{N-j_m+1, N-j_m+1} \otimes \\ E_{N-i_N+1, N-i_N+1} \otimes \dots \otimes E_{N-i_{m+1}+1, N-i_{m+1}+1}$$

we obtain that

$$\varepsilon \cdot Q_{\mathbf{i}\mathbf{j}}(u-m) = \tau_N(Q_{\mathbf{i}\mathbf{j}}(u)) A_N(u) \quad \square$$

By making use of Lemma 1.1 and Proposition 1.4 along with the equality (1.10) we obtain the following corollary to Lemma 1.5.

Corollary 1.6. *The coefficients of the series $Q_{\mathbf{i}\mathbf{j}}(u)$ commute with those of $T_{i_k j_l}(u)$ for all $k, l = 1, \dots, m$.*

Using the definitions (1.6) and (1.9) we obtain one more corollary to Lemma 1.5.

Corollary 1.7. *In the algebra $Y(\mathfrak{gl}_N)[[u^{-1}]]$ we have the equalities*

(1.14)

$$\begin{aligned}\tau_N(A_m(u)) &= A_{N-m}(u-m)/A_N(u); & m=1, \dots, N; \\ \tau_N(B_m(u)) &= -B_{N-m}(u-m)/A_N(u); & m=1, \dots, N-1;\end{aligned}$$

(1.15)

$$\begin{aligned}\tau_N(C_m(u)) &= -C_{N-m}(u-m)/A_N(u); & m=1, \dots, N-1; \\ \tau_N(D_m(u)) &= D_{N-m}(u-m)/A_N(u); & m=1, \dots, N.\end{aligned}$$

Remark. The automorphism τ_N of the algebra $Y(\mathfrak{gl}_N)$ is not involutive. Lemma 1.5 shows that

$$\tau_N^2(T_{ij}(u)) = T_{ij}(u-N) \cdot A_N(u)/A_N(u-1).$$

For more comments on the automorphism τ_N^2 of $Y(\mathfrak{gl}_N)$ see [MNO], Section 5.

Now let an arbitrary $M=0, 1, 2, \dots$ be fixed. Denote

$$\begin{aligned}\bar{\mathbf{i}} &= (1, \dots, M, M+i_1, \dots, M+i_m), \\ \bar{\mathbf{j}} &= (1, \dots, M, M+j_1, \dots, M+j_m).\end{aligned}$$

Corollary 1.8. *In the algebra $Y(\mathfrak{gl}_{M+N})[[u^{-1}]]$ we have the equality*

$$Q_{\bar{\mathbf{i}}\bar{\mathbf{j}}}(u) = \psi_{\mathbf{l}}(Q_{\mathbf{ij}}(u)) \cdot A_M(u-m) \quad \text{for} \quad \mathbf{l} = (M+1, \dots, M+N).$$

Proof. Introduce the sequences

$$\mathbf{m} = (M+i_{m+1}, \dots, M+i_N) \quad \text{and} \quad \mathbf{n} = (M+j_{m+1}, \dots, M+j_N).$$

By making use of Lemma 1.5 we then obtain the equalities

$$\begin{aligned}\psi_{\mathbf{l}}(Q_{\mathbf{ij}}(u)) &= \varepsilon \cdot \sigma_{M+N} \varphi_{\mathbf{l}} \sigma_N(Q_{\mathbf{ij}}(u)) = \\ &= \varepsilon \cdot \sigma_{M+N} \varphi_{\mathbf{l}}(Q_{\bar{\mathbf{j}}\bar{\mathbf{i}}}(-1-u)/A_N(m-1-u)) \\ &= \sigma_{M+N}(Q_{\mathbf{nm}}(-1-u) \cdot Q_{\mathbf{ll}}(m-1-u)^{-1}) \\ &= (Q_{\bar{\mathbf{i}}\bar{\mathbf{j}}}(u)/A_{M+N}(u+N-m)) \cdot (A_M(u-m)/A_{M+N}(u+N-m))^{-1} \\ &= Q_{\bar{\mathbf{i}}\bar{\mathbf{j}}}(u) \cdot A_M(u-m)^{-1} \quad \square\end{aligned}$$

Remark. By making use of Corollary 1.8 we obtain for each $m=1, \dots, N-1$ the following four equalities in the algebra $Y(\mathfrak{gl}_N)[[u^{-1}]]$:

$$\begin{aligned}A_m(u) &= A_{m-1}(u-1) \cdot \psi_{m,m+1}(T_{11}(u)), \\ B_m(u) &= A_{m-1}(u-1) \cdot \psi_{m,m+1}(T_{12}(u)), \\ C_m(u) &= A_{m-1}(u-1) \cdot \psi_{m,m+1}(T_{21}(u)), \\ D_m(u) &= A_{m-1}(u-1) \cdot \psi_{m,m+1}(T_{22}(u)).\end{aligned}$$

These equalities can be regarded as an alternative description of the Drinfeld generators for the algebra $Y(\mathfrak{gl}_N)$.

Proposition 1.9. *We have the commutation relations in $Y(\mathfrak{gl}_N)[[u^{-1}, v^{-1}]]$*

$$(1.16) \quad [A_m(u), B_n(v)] = 0 \quad \text{if } m \neq n,$$

$$(1.17) \quad [C_m(u), B_n(v)] = 0 \quad \text{if } m \neq n,$$

$$(1.18) \quad [B_m(u), B_n(v)] = 0 \quad \text{if } |m - n| \neq 1,$$

$$(1.19) \quad (u - v) \cdot [A_m(u), B_m(v)] = B_m(u) A_m(v) - B_m(v) A_m(u),$$

$$(1.20) \quad (u - v) \cdot [C_m(u), B_m(v)] = D_m(u) A_m(v) - D_m(v) A_m(u).$$

Proof. Let \mathbf{m} and \mathbf{n} be any subsequences of \mathbf{i} and \mathbf{j} respectively. Suppose that \mathbf{m} and \mathbf{n} are of the same length. Then by Corollary 1.6 the coefficients of the series $Q_{\mathbf{mn}}(u)$ commute with those of $Q_{\mathbf{ij}}(u)$. This proves (1.16) to (1.18). For the proof of the relations (1.19) to (1.20) see [NT], Section 1 \square

The proof of the next proposition is also contained in [NT], Section 1.

Proposition 1.10. *The following relation holds in the algebra $Y(\mathfrak{gl}_N)[[u^{-1}]]$:*

$$(1.21) \quad C_m(u) B_m(u - 1) = D_m(u) A_m(u - 1) - A_{m+1}(u) A_{m-1}(u - 1).$$

There is a natural Hopf algebra structure on $Y(\mathfrak{gl}_N)$ [D1]. The comultiplication $Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N)^{\otimes 2}$ is defined by the assignment of the generating series

$$(1.22) \quad T_{ij}(u) \mapsto \sum_{k=1}^N T_{ik}(u) \otimes T_{kj}(u).$$

Here and in what follows we take the tensor product over $\mathbb{C}[[u^{-1}]]$ of the elements from the algebra $Y(\mathfrak{gl}_N)[[u^{-1}]]$.

We will now consider the images of the Drinfeld generators for the algebra $Y(\mathfrak{gl}_N)$ with respect to the comultiplication

$$(1.23) \quad \Delta^{(n)} : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N)^{\otimes n}.$$

Let \mathbf{i} and \mathbf{j} be any two sequences of indices satisfying the condition (1.7).

Proposition 1.11. *We have the equality*

$$\Delta^{(n)}(Q_{\mathbf{ij}}(u)) = \sum_{\mathbf{k}^{(1)}, \mathbf{k}^{(2)}, \dots, \mathbf{k}^{(n-1)}} Q_{\mathbf{i}\mathbf{k}^{(1)}}(u) \otimes Q_{\mathbf{k}^{(1)}\mathbf{k}^{(2)}}(u) \otimes \dots \otimes Q_{\mathbf{k}^{(n-1)}\mathbf{j}}(u).$$

where $\mathbf{k}^{(1)}, \mathbf{k}^{(2)}, \dots, \mathbf{k}^{(n-1)}$ run through all the increasing sequences of the integers $1, \dots, N$ of the length m .

Proof. We employ arguments similar to those used in the proof of Lemma 1.5. Let $H \in \text{End}(\mathbb{C}^N)^{\otimes m}$ be the antisymmetrization map introduced therein. Introduce the formal power series with the coefficients in $\text{End}(\mathbb{C}^N)^{\otimes m} \otimes Y(\mathfrak{gl}_N)$

$$T_k(u) = \iota_k \otimes \text{id}(T(u)); \quad k = 1, \dots, m.$$

Then we have the relation

$$(1.24) \quad m! \cdot T_1(u) T_2(u-1) \dots T_m(u-m+1) \cdot (H \otimes 1) = \\ (H \otimes 1) \cdot T_1(u) T_2(u-1) \dots T_m(u-m+1) \cdot (H \otimes 1).$$

Put

$$I = E_{i_1 i_1} \otimes \dots \otimes E_{i_m j_m}, \quad J = E_{j_1 j_1} \otimes \dots \otimes E_{j_m j_m}, \quad K = E_{i_1 j_1} \otimes \dots \otimes E_{i_m j_m}.$$

Then by the definition (1.6) we have the equality

$$(1.25) \quad K \otimes Q_{\mathbf{ij}}(u) = (I \otimes 1) \cdot T_1(u) T_2(u-1) \dots T_m(u-m+1) \cdot (HJ \otimes 1).$$

Introduce the formal power series with coefficients in $\text{End}(\mathbb{C}^N)^{\otimes m} \otimes Y(\mathfrak{gl}_N)^{\otimes n}$

$$T_{k[s]}(u) = \iota_k \otimes \iota_s(T(u)); \quad k = 1, \dots, m; \quad s = 1, \dots, n$$

where ι_k and ι_s are respectively the embeddings $\text{End}(\mathbb{C}^N) \rightarrow \text{End}(\mathbb{C}^N)^{\otimes m}$ and $Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N)^{\otimes n}$. For any $k = 1, \dots, m$ by the definition (1.22) we then have

$$\text{id} \otimes \Delta^{(n)}(T_k(u)) = T_{k[1]}(u) T_{k[2]}(u) \dots T_{k[n]}(u).$$

Therefore from the equality (1.25) we obtain that

$$K \otimes \Delta^{(n)}(Q_{\mathbf{ij}}(u)) = (I \otimes 1) \cdot T_{1[1]}(u) T_{1[2]}(u) \dots T_{1[n]}(u) \times \\ T_{2[1]}(u-1) T_{2[2]}(u-1) \dots T_{2[n]}(u-1) \times \dots \times \\ T_{m[1]}(u-m+1) T_{m[2]}(u-m+1) \dots T_{m[n]}(u-m+1) \cdot (HJ \otimes 1)$$

where $I \otimes 1$ and $HJ \otimes 1$ are now elements of the product $\text{End}(\mathbb{C}^N)^{\otimes m} \otimes Y(\mathfrak{gl}_N)^{\otimes n}$.

Observe that the coefficients of the series $T_{k[s]}$ commute with those of $T_{l[r]}$ if $k \neq l$ and $s \neq r$. Thus the right hand side of the latter equality coincides with

$$(I \otimes 1) \cdot T_{1[1]}(u) T_{2[1]}(u-1) \dots T_{m[1]}(u-m+1) \times \\ T_{1[2]}(u) T_{2[2]}(u-1) \dots T_{m[2]}(u-m+1) \times \dots \times \\ T_{1[n]}(u) T_{2[n]}(u-1) \dots T_{m[n]}(u-m+1) \times (HJ \otimes 1).$$

By applying the relation (1.24) repeatedly we now obtain that

$$K \otimes \Delta^{(n)}(Q_{\mathbf{ij}}(u)) = (m!)^{n-1} \times \\ (I \otimes 1) \cdot T_{1[1]}(u) T_{2[1]}(u-1) \dots T_{m[1]}(u-m+1) \cdot (H \otimes 1) \times \\ T_{1[2]}(u) T_{2[2]}(u-1) \dots T_{m[2]}(u-m+1) \cdot (H \otimes 1) \times \dots \times \\ T_{1[n]}(u) T_{2[n]}(u-m+1) \dots T_{m[n]}(u-m+1) \cdot (HJ \otimes 1).$$

Proposition 1.11 follows from this equality and from the relation (1.24) \square

Corollary 1.12. *We have the equality $\Delta^{(n)}(A_N(u)) = (A_N(u))^{\otimes n}$.*

Note that $\Delta^{(n)}(A_m(u)) \neq (A_m(u))^{\otimes n}$ in general. However, the next corollary will be sufficient for our purposes. Endow the algebra $Y(\mathfrak{gl}_N)$ with the \mathbb{Z} -grading \deg determined by

$$(1.26) \quad \deg T_{ij}^{(s)} = i - j; \quad s = 0, 1, 2, \dots;$$

see the relations (1.1). We will extend this grading to the algebra $Y(\mathfrak{gl}_N)[[u^{-1}]]$ by assuming that $\deg u^{-1} = 0$. The definitions (1.6) and (1.9) then show that

$$\deg A_m(u) = 0, \quad \deg B_m(u) = -1, \quad \deg C_m(u) = 1.$$

The algebras $Y(\mathfrak{gl}_N)^{\otimes n}$ and $Y(\mathfrak{gl}_N)^{\otimes n}[[u^{-1}]]$ then acquire grading by the group \mathbb{Z}^n . We will fix the lexicographical ordering on the group \mathbb{Z}^n .

Corollary 1.13. *For every $m = 1, \dots, N$ we have the equality*

$$\Delta^{(n)}(A_m(u)) = (A_m(u))^{\otimes n} + \text{terms of the smaller degrees.}$$

Proof. Let us apply Lemma 1.11 to $\mathbf{i} = \mathbf{j} = (1, \dots, m)$. Then $\deg Q_{\mathbf{i}\mathbf{k}^{(1)}}(u) < 0$ unless $\mathbf{k}^{(1)} = \mathbf{i}$. Repeating this argument we obtain the required statement \square

In Section 4 we will make use of the following observation. For each permutation g of $1, 2, \dots, N$ denote by $A_g(\mathfrak{gl}_N)$ the subalgebra in $Y(\mathfrak{gl}_N)$ generated by all the coefficients of the series $Q_{\mathbf{i}\mathbf{i}}(u)$ where

$$(1.27) \quad \mathbf{i} = (1, \dots, N) \setminus g(m+1), \dots, g(N); \quad m = 1, \dots, N.$$

Proposition 1.14. *Let V be any finite-dimensional module of the algebra $Y(\mathfrak{gl}_N)$. The images in $\text{End}(V)$ of all the subalgebras $A_g(\mathfrak{gl}_N)$ are conjugated to each other.*

Proof. Denote by θ the action of the algebra $Y(\mathfrak{gl}_N)$ in V . Take the embedding of the algebra $U(\mathfrak{gl}_N)$ into $Y(\mathfrak{gl}_N)$ defined by $E_{ji} \mapsto T_{ij}^{(1)}$. By the relations (1.1) we have

$$[E_{ji}, T_{kl}^{(s)}] = \delta_{il} \cdot T_{kj}^{(s)} - \delta_{kj} \cdot T_{il}^{(s)}; \quad s = 1, 2, \dots$$

Therefore the image of \mathfrak{gl}_N in $\text{End}(V)$ contains an element G such that

$$\exp G \cdot \theta(T_{ij}^{(s)}) = \theta(T_{g(i)g(j)}^{(s)}) \cdot \exp G; \quad s = 1, 2, \dots$$

The latter equalities along with (1.12) imply that for each $m = 1, \dots, N$ we have

$$\exp G \cdot \theta(A_m(u)) = \theta(Q_{\mathbf{i}\mathbf{i}}(u)) \cdot \exp G$$

where the increasing sequence \mathbf{i} is defined in (1.27) \square

2. Modules with Gelfand-Zetlin bases

In this section we will consider certain family of finite-dimensional modules over the algebra $Y(\mathfrak{gl}_N)$. Some of them will be irreducible and have a simple spectrum

with respect to the action of the maximal commutative subalgebra $A(\mathfrak{gl}_N)$. The eigenbases of $A(\mathfrak{gl}_N)$ in the latter modules will be called Gelfand-Zetlin bases. We will determine the eigenvalues of the coefficients of the series $A_1(u), \dots, A_N(u)$ corresponding to the vectors of these bases. The action of the remaining Drinfeld generators of the algebra $Y(\mathfrak{gl}_N)$ on the vectors of these bases will be described in Section 3.

Consider the matrix units E_{ij} with $i, j = 1, \dots, N$ as generators of the Lie algebra \mathfrak{gl}_N . For every non-increasing sequence of integers $\lambda = (\lambda_1, \dots, \lambda_N)$ denote by V_λ the irreducible \mathfrak{gl}_N -module of the highest weight λ . If $\xi \in V_\lambda$ is the highest weight vector then we have $E_{ii} \cdot \xi = \lambda_i \xi$ and $E_{ij} \cdot \xi = 0$ for $i < j$.

Denote by \mathcal{T}_λ the set of all arrays Λ with integral entries of the form

$$\begin{array}{ccccccc} \lambda_{N1} & \lambda_{N2} & \dots & \dots & \dots & \dots & \lambda_{NN} \\ & \lambda_{N-1,1} & \lambda_{N-1,2} & \dots & \dots & \dots & \lambda_{N-1,N-1} \\ & & \ddots & & \ddots & & \\ & & & \lambda_{21} & \lambda_{22} & & \\ & & & & \lambda_{11} & & \end{array}$$

where $\lambda_{Ni} = \lambda_i$ and $\lambda_{mi} \leq \lambda_i$ for all i and m . An array $\Lambda \in \mathcal{T}_\lambda$ is called a *Gelfand-Zetlin scheme* if $\lambda_{mi} \geq \lambda_{m-1,i} \geq \lambda_{m,i+1}$ for all possible m and i . Denote by \mathcal{S}_λ the subset in \mathcal{T}_λ consisting of the Gelfand-Zetlin schemes.

There is a canonical decomposition of the space V_λ into the direct sum of one-dimensional subspaces associated with the ascending chain of subalgebras

$$(2.1) \quad U(\mathfrak{gl}_1) \subset U(\mathfrak{gl}_2) \subset \dots \subset U(\mathfrak{gl}_N)$$

determined by the standard embeddings. These subspaces are parametrized by the elements of the set \mathcal{S}_λ . For each $m = 1, 2, \dots, N-1$ the subspace $V_\Lambda \subset V_\lambda$ corresponding to the scheme $\Lambda \in \mathcal{S}_\lambda$ is contained in an irreducible \mathfrak{gl}_m -submodule of the highest weight $(\lambda_{m1}, \lambda_{m2}, \dots, \lambda_{mm})$. These conditions define the subspace V_Λ uniquely, cf. [GZ]. The highest weight subspace in V_λ corresponds to the scheme Λ where $\lambda_{mi} = \lambda_i$ for every $m = 1, \dots, N$.

Consider the \mathbb{Z} -grading on the algebra $U(\mathfrak{gl}_N)$ such that the degree of the element E_{ij} equals $j - i$. It is the grading by the eigenvalues of the adjoint action in $U(\mathfrak{gl}_N)$ of the element $NE_{11} + (N-1)E_{22} + \dots + E_{NN}$. Endow the space V_λ with the \mathbb{Z} -grading by the eigenvalues of the action of the same element. Then the action of $U(\mathfrak{gl}_N)$ in the space V_λ becomes a graded action. Note that every subspace V_Λ in V_λ is a weight subspace and hence homogeneous; its degree equals

$$(2.2) \quad \sum_{m=1}^N (N-m+1) \left(\sum_{i=1}^m \lambda_{mi} - \sum_{i=1}^{m-1} \lambda_{m-1,i} \right) = \sum_{m=1}^N \sum_{i=1}^m \lambda_{mi}.$$

Let us now make use of the homomorphism $\pi_N : Y(\mathfrak{gl}_N) \rightarrow U(\mathfrak{gl}_N)$ defined by (1.6). Note that this homomorphism preserves the \mathbb{Z} -grading on the algebra $Y(\mathfrak{gl}_N)$ introduced in the end of Section 1. For each $m = 1, \dots, N$ and any $\Lambda \in \mathcal{T}_\lambda$ introduce the rational function

$$(2.3) \quad \alpha_{m\Lambda}(u) = \prod_{i=1}^m \frac{(u + \lambda_{mi} - i + 1)}{(u - i + 1)};$$

note that $\alpha_{m\Lambda}(u)$ can be also regarded as a formal power series in u^{-1} .

Lemma 2.1. *The subspace $V_\Lambda \subset V_\lambda$ is an eigenspace for the coefficients of the series $\pi_N(A_m(u))$, the eigenvalues are the respective coefficients of the series $\alpha_{m\Lambda}(u)$.*

Proof. By the definition (1.6) we have

$$\pi_N(A_m(u)) \in \prod_{i=1}^m \frac{(u + E_{ii} - i + 1)}{(u - i + 1)} + U(\mathfrak{gl}_m)[[u^{-1}]] \cdot \mathfrak{n}_m$$

where \mathfrak{n}_m is the subalgebra in \mathfrak{gl}_m spanned by the elements E_{ij} with $i < j$. Due to Proposition 1.4 the coefficients of the series $\pi_N(A_m(u))$ belong to the center of the algebra $U(\mathfrak{gl}_m)$ and act in any irreducible \mathfrak{gl}_m -submodule of V_λ via scalars. Applying these coefficients to the highest weight vector in an irreducible \mathfrak{gl}_m -submodule of V_λ we get the statement of Lemma 2.1 by the definition of the subspace V_Λ \square

Note that due to the conditions $\lambda_{m1} \geq \lambda_{m2} \geq \dots \geq \lambda_{mm}$ the scheme $\Lambda \in \mathcal{S}_\lambda$ can be uniquely restored from the collection of rational functions $\alpha_{1\Lambda}(u), \dots, \alpha_{N\Lambda}(u)$. Therefore the action in V_λ of the commutative subalgebra $\pi_N(A(\mathfrak{gl}_N))$ in $U(\mathfrak{gl}_N)$ has a simple spectrum.

Remark. The subalgebra $\pi_N(A(\mathfrak{gl}_N))$ in $U(\mathfrak{gl}_N)$ coincides with the subalgebra generated by the centres of all the algebras in (2.1). The latter subalgebra in $U(\mathfrak{gl}_N)$ is maximal commutative [C], Theorem 2.2.

Let M run through the set $\{0, 1, 2, \dots\}$. Fix the standard embedding of the algebra $U(\mathfrak{gl}_M)$ into $U(\mathfrak{gl}_{M+N})$. For any pair of non-increasing sequences of integers

$$(2.4) \quad \lambda = (\lambda_1, \dots, \lambda_M, \lambda_{M+1}, \dots, \lambda_{M+N}) \quad \text{and} \quad \mu = (\mu_1, \dots, \mu_M)$$

denote by $V_{\lambda, \mu}$ the subspace in the \mathfrak{gl}_{M+N} -module V_λ formed by all the singular vectors with respect to \mathfrak{gl}_M of the weight μ . Denote by $\mathcal{S}_{\lambda, \mu}$ the subset in \mathcal{S}_λ formed by all the arrays

$$\Lambda = (\lambda_{li} \mid 1 \leq i \leq l \leq M + N)$$

such that $\lambda_{li} = \mu_i$ for every $l = 1, \dots, M$. The space $V_{\lambda, \mu}$ is the direct sum of the subspaces V_Λ in V_λ where Λ runs through the set $\mathcal{S}_{\lambda, \mu}$. We will assume that the set $\mathcal{S}_{\lambda, \mu}$ is not empty.

The action in V_λ of the centralizer of the subalgebra $U(\mathfrak{gl}_M)$ in $U(\mathfrak{gl}_{M+N})$ preserves the subspace $V_{\lambda, \mu}$ by the definition of this subspace. By Proposition 1.3 the image of the homomorphism

$$\pi_{M+N} \psi_1 : Y(\mathfrak{gl}_N) \rightarrow U(\mathfrak{gl}_{M+N}); \quad \mathbf{l} = (M + 1, \dots, M + N)$$

commutes with the subalgebra $U(\mathfrak{gl}_M)$ in $U(\mathfrak{gl}_{M+N})$. We will regard $V_{\lambda, \mu}$ as a module over the Yangian $Y(\mathfrak{gl}_N)$ by making use of this homomorphism. Introduce the rational function

$$\alpha_\mu(u) = \prod_{i=1}^M \frac{(u + \mu_i - i + 1)}{(u - i + 1)};$$

it can be also regarded as a formal power series in u^{-1} . We have $\alpha_{M, \Lambda}(u) = \alpha_\mu(u)$ for every $\Lambda \in \mathcal{S}_{\lambda, \mu}$.

Proposition 2.2. *Suppose that $m \in \{1, \dots, N\}$ and $\Lambda \in \mathcal{S}_{\lambda, \mu}$. The subspace $V_\Lambda \subset V_{\lambda, \mu}$ is an eigenspace for the coefficients of the series $A_m(u)$, the eigenvalues being the respective coefficients of the series $\alpha_{M+m, \Lambda}(u)/\alpha_\mu(u-m)$.*

Proof. Let us apply Corollary 1.8 to $\mathbf{i} = \mathbf{j} = (1, \dots, m)$. Then we get the equality

$$\psi_1(A_m(u)) = A_{M+m}(u)/A_M(u-m); \quad \mathbf{l} = (M+1, \dots, M+N)$$

in the algebra $Y(\mathfrak{gl}_{M+N})[[u^{-1}]]$. By applying to this equality the homomorphism π_{M+N} and then making use of Lemma 2.1 we obtain the required statement \square

The space $V_{\lambda, \mu}$ inherits from V_λ the \mathbb{Z} -grading. Consider the \mathbb{Z} -grading \deg on the algebra $Y(\mathfrak{gl}_N)$ defined by (1.26).

Proposition 2.3. *The action of the algebra $Y(\mathfrak{gl}_N)$ in the space $V_{\lambda, \mu}$ is graded.*

Proof. The action of the algebra $U(\mathfrak{gl}_{M+N})$ in the space V_λ is a graded action. The homomorphism $\pi_{M+N} : Y(\mathfrak{gl}_{M+N}) \rightarrow U(\mathfrak{gl}_{M+N})$ preserves the \mathbb{Z} -grading. But the embedding $\psi_1 : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_{M+N})$ with $\mathbf{l} = (M+1, \dots, M+N)$ also preserves the \mathbb{Z} -grading. Indeed, we have $\psi_1 = \sigma_{M+N} \varphi_1 \sigma_M$. Here the embedding $\varphi_1 : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_{M+N})$ preserves \deg by definition. But due to (1.3) we have $\deg \cdot \sigma_N = -\deg$ and similarly $\deg \cdot \sigma_{M+N} = -\deg$ \square

Denote by Λ_0 the array $(\kappa_{li} \mid 1 \leq i \leq l \leq M+N)$ where

$$(2.5) \quad \kappa_{li} = \begin{cases} \mu_i & \text{if } l < M, \\ \min(\lambda_i, \mu_{i-l+M}) & \text{if } l \geq M \text{ and } i > l-M, \\ \lambda_i & \text{if } l > M \text{ and } i \leq l-M. \end{cases}$$

Lemma 2.4. *We have $\Lambda_0 \in \mathcal{S}_{\lambda, \mu}$.*

Proof. For every $i \leq N$ we have $\kappa_{M+N, i} = \lambda_i$ by definition. Since the set $\mathcal{S}_{\lambda, \mu}$ is non-empty, we have the inequality $\lambda_i \leq \mu_{i-N}$ for every $i > N$. Therefore $\kappa_{M+N, i} = \lambda_i$ for $i > N$ also. Similarly, for every $i \leq M$ we have $\lambda_i \geq \mu_i$ and $\kappa_{Mi} = \mu_i$. Since the sequences λ and μ are non-increasing, we have the inequalities $\kappa_{li} \geq \kappa_{l-1, i} \geq \kappa_{l, i-1}$ for all possible l and i \square

Observe that for any scheme $\Lambda \in \mathcal{S}_{\lambda, \mu}$ we have the inequalities $\lambda_{li} \leq \kappa_{li}$ for all l and i . Therefore the subspace V_{Λ_0} in $V_{\lambda, \mu}$ has the maximal degree, see (2.2). Thus we obtain the following corollary to Proposition 2.3.

Corollary 2.5. *The subspace $V_{\Lambda_0} \subset V_{\lambda, \mu}$ is annihilated by the action of any element in $Y(\mathfrak{gl}_N)$ of a positive degree.*

We will denote by $\mathcal{T}_{\lambda, \mu}$ the subset in \mathcal{T}_λ formed by the arrays Λ such that $\lambda_{li} = \mu_i$ for each $l = 1, \dots, M$ and $\lambda_{li} \leq \kappa_{li}$ for all l and i . We have $\alpha_{M, \Lambda}(u) = \alpha_\mu(u)$ for every $\Lambda \in \mathcal{T}_{\lambda, \mu}$.

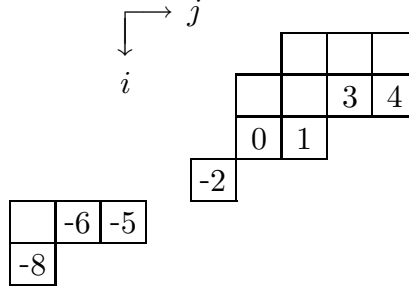
Let $\beta = (\beta_1, \beta_2, \dots)$ and $\gamma = (\gamma_1, \gamma_2, \dots)$ be any two non-increasing sequences of integers such that $\beta_i \geq \gamma_i$ for each $i = 1, 2, \dots$ and $\beta_i = \gamma_i$ for every i large enough. The skew Young diagram corresponding to the sequences β, γ is the set

$$\{(i, j) \in \mathbb{Z}^2 \mid i \geq 1, \beta_i \geq j > \gamma_i\}$$

We shall employ the usual graphic representation of a diagram: a point $(i, j) \in \mathbb{Z}^2$ is represented by the unit box on the plane \mathbb{R}^2 with the centre (i, j) , the coordinates i and j on \mathbb{R}^2 increasing from top to bottom and from left to right respectively. The *content* of the box corresponding to (i, j) is the difference $j - i$. Here is the diagram corresponding to the sequences

$$\beta = (6, 6, 4, 2, 0, -2, -3, -3, \dots) \quad \text{and} \quad \gamma = (3, 2, 2, 1, -3, -3, \dots);$$

in this diagram we have indicated the content of the bottom box for every column.



Now consider the scheme $\Lambda_0 \in \mathcal{S}_{\lambda, \mu}$. We have $\kappa_{M+m, M+m} \geq \kappa_{M+N, M+N} = \lambda_{M+N}$ and $\kappa_{M+m, i} \geq \kappa_{Mi}$ for all possible m and i . For every $m = 0, 1, \dots, N$ denote by $\kappa^{(m)}$ the skew Young diagram corresponding to the sequences

$$(\kappa_{M+m, 1}, \kappa_{M+m, 2}, \dots, \kappa_{M+m, M+m}, \lambda_{M+N}, \lambda_{M+N} \dots),$$

$$(\kappa_{M1}, \kappa_{M2}, \dots, \kappa_{MM}, \lambda_{M+N}, \lambda_{M+N} \dots).$$

Thus the diagram $\kappa^{(0)}$ is empty. The diagram $\kappa^{(N)}$ will be denoted by λ/μ .

Lemma 2.6. *Any column of the diagram $\kappa^{(m)}$ consists of at most m boxes. For $m \geq 1$ the diagram $\kappa^{(m-1)}$ is obtained from $\kappa^{(m)}$ by removing the bottom box from every column of the height m .*

Proof. If $M = 0$ then both $\kappa^{(m)}$ and $\kappa^{(m-1)}$ are usual Young diagrams. The latter diagram is obtained from the former by removing the row m . Lemma 2.6 is then evidently true. We will suppose that $M \geq 1$. If the top and the bottom boxes of a column in $\kappa^{(m)}$ are (i, j) and $(i+l, j)$ respectively then $\kappa_{Mi} < j \leq \kappa_{M+m, i+l}$. But for every $l \geq m$ we have the inequalities $\kappa_{Mi} \geq \kappa_{M+m, i+m} \geq \kappa_{M+m, i+l}$ since $\Lambda_0 \in \mathcal{S}_{\lambda, \mu}$. This proves the first statement of Lemma 2.6.

To prove the second statement observe that $\kappa_{M+m-1, i}$ differs from $\kappa_{M+m, i}$ only if $i \geq m \geq 1$ and $\kappa_{M+m, i} > \kappa_{M, i-m+1}$. In this case we have $\kappa_{M+m-1, i} = \kappa_{M, i-m+1}$. Each of the columns corresponding to j with $\kappa_{M, i-m+1} < j \leq \kappa_{M+m, i}$ has the height m . The box (i, j) is then the bottom box of this column. The row i of $\kappa^{(m-1)}$ is obtained from that of $\kappa^{(m)}$ by removing each of these boxes \square

Note that the condition $\mathcal{S}_{\lambda, \mu} \neq \emptyset$ on the sequences (2.4) is equivalent to the condition that any column of the diagram λ/μ consists of at most N boxes. Now for every $m = 1, \dots, N-1$ introduce the polynomial in u

$$(2.6) \quad P_{m, \lambda/\mu}(u) = \prod_k (u + k)$$

where k runs through the collection of the contents of the bottom boxes in the columns of the height m in the diagram λ/μ . We will make use of the following observation.

Proposition 2.7. *For every $m = 1, \dots, N - 1$ we have the equality*

$$\frac{\alpha_{M+m+1, \Lambda_0}(u) \alpha_{M+m-1, \Lambda_0}(u-1)}{\alpha_{M+m, \Lambda_0}(u) \alpha_{M+m, \Lambda_0}(u-1)} = \frac{P_{m, \lambda/\mu}(u-1)}{P_{m, \lambda/\mu}(u)}.$$

Proof. For every $m = 0, \dots, N$ introduce the polynomial

$$Q_m(u) = \prod_k (u + k)$$

where k runs through the collection of the contents of the bottom boxes in the columns of the height m in the diagram $\kappa^{(m)}$. Then by Lemma 2.6 we have

$$(2.7) \quad P_{m, \lambda/\mu}(u) = Q_m(u) / Q_{m+1}(u+1); \quad m = 1, \dots, N-1.$$

On the other hand, by the definition (2.3) and again by Lemma 2.6 we have

$$(2.8) \quad \frac{\alpha_{M+m, \Lambda_0}(u)}{\alpha_{M+m-1, \Lambda_0}(u)} = (u - M - m + 1) \cdot \frac{Q_m(u+1)}{Q_m(u)}; \quad m = 1, \dots, N.$$

By comparing (2.7) and (2.8) we obtain the required statement \square

The relations (1.2) imply that for any $h \in \mathbb{C}$ the assignment $T_{ij}(u) \mapsto T_{ij}(u+h)$ determines an automorphism of the algebra $Y(\mathfrak{gl}_N)$; here the series in $(u+h)^{-1}$ should be re-expanded in u^{-1} . We will denote by $V_{\lambda, \mu}(h)$ the $Y(\mathfrak{gl}_N)$ -module obtained from $V_{\lambda, \mu}$ by the pullback through this automorphism.

Let us now make use of the comultiplication (1.23). For every $s = 1, \dots, n$ fix some $h^{(s)} \in \mathbb{C}$ along with a pair of non-increasing sequences of integers

$$\lambda^{(s)} = (\lambda_1^{(s)}, \dots, \lambda_{M^{(s)}}^{(s)}, \lambda_{M^{(s)}+1}^{(s)}, \dots, \lambda_{M^{(s)}+N}^{(s)}) \quad \text{and} \quad \mu^{(s)} = (\mu_1^{(s)}, \dots, \mu_{M^{(s)}}^{(s)})$$

where $M^{(s)} \in \{0, 1, 2, \dots\}$. Suppose that each of the sets $S_{\lambda^{(s)}, \mu^{(s)}}$ is not empty. Consider the $Y(\mathfrak{gl}_N)$ -module

$$(2.9) \quad W = V_{\lambda^{(1)}, \mu^{(1)}}(h^{(1)}) \otimes \dots \otimes V_{\lambda^{(n)}, \mu^{(n)}}(h^{(n)}).$$

In Section 3 we will prove that the $Y(\mathfrak{gl}_N)$ -module W is irreducible provided $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$. Theorem 2.9 below is contained in [C], Theorem 2.6. However, we will give here the proof.

For each $m = 1, \dots, N$ and every collection of schemes $\Lambda^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ where $s = 1, \dots, n$ introduce the rational function

$$(2.10) \quad \chi_{m, \Lambda^{(1)}, \dots, \Lambda^{(n)}}(u) = \prod_{s=1}^n \frac{\alpha_{M^{(s)}+m, \Lambda^{(s)}}(u + h^{(s)})}{\alpha_{\mu^{(s)}}(u - m + h^{(s)})};$$

it can be also regarded as a formal power series in u^{-1} .

Proposition 2.8. *For any collection of schemes $\Lambda^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ with $s = 1, \dots, n$ there exists a non-zero vector ξ in W such that for every $m = 1, \dots, N$ we have the equality*

$$A_m(u) \cdot \xi = \xi \cdot \chi_{m, \Lambda^{(1)}, \dots, \Lambda^{(n)}}(u).$$

Proof. We will derive this statement from Corollary 1.13. Each of the vector spaces $V_{\lambda^{(1)}, \mu^{(1)}}, \dots, V_{\lambda^{(n)}, \mu^{(n)}}$ is \mathbb{Z} -graded. Their tensor product acquires grading by the group \mathbb{Z}^n ; we have fixed the lexicographical ordering on the group \mathbb{Z}^n .

For each $s = 1, \dots, n$ and every $\Lambda^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ choose a non-zero vector $\xi_{\Lambda^{(s)}}$ in the subspace $V_{\Lambda^{(s)}} \subset V_{\lambda^{(s)}, \mu^{(s)}}$. The vectors $\xi_{\Lambda^{(1)}} \otimes \dots \otimes \xi_{\Lambda^{(n)}}$ form a homogeneous basis in W . By Corollary 1.13 and Proposition 2.3 for every $m = 1, \dots, N$ we have the equality of formal series in u^{-1}

$$A_m(u) \cdot \xi_{\Lambda^{(1)}} \otimes \dots \otimes \xi_{\Lambda^{(n)}} = \xi_{\Lambda^{(1)}} \otimes \dots \otimes \xi_{\Lambda^{(n)}} \cdot \chi_{m, \Lambda^{(1)}, \dots, \Lambda^{(n)}}(u) + \Xi(u)$$

where $\Xi(u)$ involves only the basic vectors in W with the degrees smaller than that of $\xi_{\Lambda^{(1)}} \otimes \dots \otimes \xi_{\Lambda^{(n)}}$. Proposition 2.8 now follows \square

Theorem 2.9. *Suppose that $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$. Then there is a basis*

$$(\xi_{\Lambda^{(1)}, \dots, \Lambda^{(n)}} \mid \Lambda^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}; s = 1, \dots, n)$$

in the space W such that for every $m = 1, \dots, N$ we have the equality

$$(2.11) \quad A_m(u) \cdot \xi_{\Lambda^{(1)}, \dots, \Lambda^{(n)}} = \xi_{\Lambda^{(1)}, \dots, \Lambda^{(n)}} \cdot \chi_{m, \Lambda^{(1)}, \dots, \Lambda^{(n)}}(u).$$

The equalities (2.11) determine the vector $\xi_{\Lambda^{(1)}, \dots, \Lambda^{(n)}}$ uniquely up to a scalar factor.

Proof. Since $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$, the collection of the schemes

$$\Lambda^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}; \quad s = 1, \dots, n$$

can be uniquely restored from the the collection of the rational functions

$$\chi_{m, \Lambda^{(1)}, \dots, \Lambda^{(n)}}(u); \quad m = 1, \dots, N.$$

Thus the action in W of the commutative algebra $A(\mathfrak{gl}_N)$ has a simple spectrum by Proposition 2.8. The vector $\xi_{\Lambda^{(1)}, \dots, \Lambda^{(n)}} \in W$ can be determined as an eigenvector of this action satisfying the equality (2.11) for every $m = 1, \dots, N$ \square

Let V be an irreducible finite-dimensional module of the algebra $Y(\mathfrak{gl}_N)$. A basis in the vector space V consisting of the eigenvectors of the subalgebra $A(\mathfrak{gl}_N)$ in $Y(\mathfrak{gl}_N)$ will be called a *Gelfand-Zelin basis*. In Section 4 we will demonstrate that any module V which admits such a basis, can be obtained from some module W by applying to the algebra $Y(\mathfrak{gl}_N)$ an automorphism of the form ω_f .

A non-zero vector $\xi \in V$ is called *singular* if it is annihilated by all the coefficients of the series $C_1(u), \dots, C_{N-1}(u)$. The vector ξ is then unique up to a scalar multiplier and is an eigenvector for the coefficients of the series $A_1(u), \dots, A_N(u)$; see [D2], Theorem 2. Moreover, then

$$(2.12) \quad \frac{A_{m+1}(u) A_{m-1}(u-1)}{A_m(u) A_m(u-1)} \cdot \xi = \frac{P_m(u-1)}{P_m(u)} \cdot \xi; \quad m = 1, \dots, N-1$$

for certain monic polynomials $P_1(u), \dots, P_{N-1}(u)$ with the coefficients in \mathbb{C} . These $N-1$ polynomials are called the *Drinfeld polynomials* of the module V . Every collection of $N-1$ monic polynomials arises in this way. The modules with the same Drinfeld polynomials may differ only by an automorphism of the algebra $Y(\mathfrak{gl}_N)$ of the form ω_f . The following properties [D2] of the vector ξ will be used in Section 4.

Proposition 2.10. *The vector ξ is annihilated by every coefficient of the series $T_{ij}(u)$ with $i > j$. Furthermore, we have*

$$\frac{T_{m+1,m+1}(u-m)}{T_{m,m}(u-m)} \cdot \xi = \frac{P_m(u-1)}{P_m(u)} \cdot \xi; \quad m = 1, \dots, N-1.$$

Denote by \overline{V} the irreducible $Y(\mathfrak{gl}_N)$ -module obtained from V by the pullback through the automorphism τ_N . Let $\overline{P}_1(u), \dots, \overline{P}_{N-1}(u)$ be the Drinfeld polynomials of the module \overline{V} . The next proposition will be also used in Section 4.

Proposition 2.11. *For each $m = 1, \dots, N-1$ we have $\overline{P}_m(u) = P_{N-m}(u-m)$.*

Proof. By the equalities (1.14) the automorphism τ_N preserves the subalgebra in $Y(\mathfrak{gl}_N)$ generated by the coefficients of the series $C_1(u), \dots, C_{N-1}(u)$ and $A_N(u)$. Therefore V and \overline{V} have the same singular vectors. The required statement now follows from the equalities (1.15) and (2.12). \square

In the remainder of this section we will point out a singular vector in the module W and evaluate the corresponding Drinfeld polynomials provided W is irreducible. For every $s = 1, \dots, n$ determine the scheme

$$\Lambda_0^{(s)} = (\kappa_{li}^{(s)} \mid 1 \leq i \leq l \leq M^{(s)} + N) \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$$

in the way analogous to (2.5): we put

$$\kappa_{li}^{(s)} = \begin{cases} \mu_i^{(s)} & \text{if } l < M^{(s)}, \\ \min(\lambda_i^{(s)}, \mu_{i-l+M^{(s)}}^{(s)}) & \text{if } l \geq M^{(s)} \text{ and } i > l - M^{(s)}, \\ \lambda_i^{(s)} & \text{if } l > M^{(s)} \text{ and } i \leq l - M^{(s)}. \end{cases}$$

Choose a non-zero vector $\xi_{\Lambda_0^{(s)}}$ in the subspace $V_{\Lambda_0^{(s)}} \subset V_{\lambda^{(s)}, \mu^{(s)}}$. Consider the vector $\xi_0 = \xi_{\Lambda_0^{(1)}} \otimes \dots \otimes \xi_{\Lambda_0^{(n)}}$ in the space W .

Proposition 2.12. *The vector ξ_0 in W is annihilated by all the coefficients of the series $C_1(u), \dots, C_{N-1}(u)$. For each $m = 1, \dots, N$ we have the equality*

$$(2.13) \quad A_m(u) \cdot \xi_0 = \xi_0 \cdot \chi_{m, \Lambda_0^{(1)}, \dots, \Lambda_0^{(n)}}(u).$$

Proof. Denote by $X^{(n)}$ the vector subspace in $Y(\mathfrak{gl}_N)^{\otimes n}[[u^{-1}]]$ spanned by all homogeneous elements whose degree in \mathbb{Z}^n has at least one positive component. Since $\deg C_m(u) = 1$, by the definition of the comultiplication (1.23) we have

$$\Delta^{(n)}(C_m(u)) \in X^{(n)}; \quad m = 1, \dots, N-1.$$

Therefore $C_m(u) \cdot \xi_0 = 0$ by Corollary 2.5. Furthermore, by Corollary 1.13 we have

$$\Delta^{(n)}(A_m(u)) \in (A_m(u))^{\otimes n} + X^{(n)}; \quad m = 1, \dots, N-1.$$

Thus by Proposition 2.2 the vector ξ_0 satisfies (2.13) for every $m = 1, \dots, N$ \square

We now obtain the following corollary to Propositions 2.7 and 2.12.

Corollary 2.13. *Suppose that the module W is irreducible. Then the vector ξ_0 in W is singular. The Drinfeld polynomials of the module W are*

$$(2.14) \quad \prod_{s=1}^n P_{m, \lambda^{(s)}/\mu^{(s)}}(u + h^{(s)}); \quad m = 1, \dots, N-1.$$

This fact will be substantially used in Section 4.

3. Action of the Drinfeld generators

In this section we will keep fixed the sequences $\lambda^{(1)}, \mu^{(1)}, \dots, \lambda^{(n)}, \mu^{(n)}$ along with the parameters $h^{(1)}, \dots, h^{(n)} \in \mathbb{C}$. From now on until the end of this section we will be assuming that $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$. For each $m = 0, \dots, N$ denote

$$\begin{aligned} \rho_m(u) &= \prod_{s=1}^n \left(\alpha_{\mu^{(s)}}(u - m + h^{(s)}) \prod_{i=1}^{M^{(s)}+m} (u - i + 1 + h^{(s)}) \right) \\ &= \prod_{s=1}^n \left(\prod_{i=1}^m (u - i + 1 + h^{(s)}) \prod_{i=1+m}^{M^{(s)}+m} (u + \mu_{i-m}^{(s)} - i + 1 + h^{(s)}) \right); \end{aligned}$$

it is a polynomial in u of the degree

$$r_m = M^{(1)} + \dots + M^{(n)} + mn.$$

Note that for each $m = 1, \dots, N-1$ we have the equality

$$(3.1) \quad \rho_m(u) \rho_m(u-1) = \rho_{m+1}(u) \rho_{m-1}(u-1).$$

Consider the action of the coefficients of the formal Laurent series in u^{-1}

$$(3.2) \quad \rho_m(u) A_m(u) \quad \text{and} \quad \rho_m(u) B_m(u), \quad \rho_m(u) C_m(u), \quad \rho_m(u) D_m(u)$$

in the $Y(\mathfrak{gl}_N)$ -module W . Denote by $a_m(u)$ and $b_m(u), c_m(u), d_m(u)$ the series with the coefficients in $\text{End}(W)$ corresponding to the series (3.2). Observe that $a_m(u)$ is a polynomial in u by Theorem 2.9.

Proposition 3.1. *For every $m = 1, \dots, N-1$ these $b_m(u), c_m(u)$ and $d_m(u)$ are polynomial in u . The degrees of $b_m(u)$ and $c_m(u)$ do not exceed $r_m - 1$.*

Proof. Due to Proposition 1.11 each of the images

$$\Delta^{(n)}(B_m(u)), \Delta^{(n)}(C_m(u)), \Delta^{(n)}(D_m(u)) \in Y(\mathfrak{gl}_N)[[u^{-1}]]$$

is a finite sum of the series of the form

$$Q_{\mathbf{k}^{(0)}\mathbf{k}^{(1)}}(u) \otimes Q_{\mathbf{k}^{(1)}\mathbf{k}^{(2)}}(u) \otimes \dots \otimes Q_{\mathbf{k}^{(n-1)}\mathbf{k}^{(n)}}(u)$$

where $\mathbf{k}^{(0)}, \mathbf{k}^{(1)}, \dots, \mathbf{k}^{(n)}$ are increasing sequences of the integers $1, \dots, N$ of the length m . For each $\mathbf{k}^{(s)} = (k_1^{(s)}, \dots, k_m^{(s)})$ denote

$$\bar{\mathbf{k}}^{(s)} = (1, \dots, M^{(s)}, M^{(s)} + k_1^{(s)}, \dots, M^{(s)} + k_m^{(s)}).$$

Due to Corollary 1.8 for each $s = 1, \dots, n$ the action of the coefficients of the series $Q_{\mathbf{k}^{(s-1)}\mathbf{k}^{(s)}}(u)$ in the module $V_{\lambda^{(s)}, \mu^{(s)}}(h^{(s)})$ can be defined as that of the coefficients of

$$\pi_{M^{(s)}+N}(Q_{\mathbf{k}^{(s-1)}\mathbf{k}^{(s)}}(u + h^{(s)}) \cdot A_M^{(s)}(u - m + h^{(s)})^{-1}) \in U(\mathfrak{gl}_{M^{(s)}+N})[[u^{-1}]]$$

in the subspace $V_{\lambda^{(s)}, \mu^{(s)}} \subset V_{\lambda^{(s)}}$. But the product

$$\pi_{M^{(s)}+N}(Q_{\mathbf{k}^{(s-1)}\mathbf{k}^{(s)}}(u + h^{(s)})) \cdot \prod_{i=1}^{M^{(s)}+m} (u - i + 1 + h^{(s)})$$

is a polynomial in u by the definition of the homomorphism $\pi_{M^{(s)}+N}$. By Lemma 2.1 the coefficients of

$$\pi_{M^{(s)}+N}(A_M^{(s)}(u - m + h^{(s)})) \in U(\mathfrak{gl}_{M^{(s)}+N})[[u^{-1}]]$$

act in $V_{\lambda^{(s)}, \mu^{(s)}}$ as the respective coefficients of the series $\alpha_{\mu^{(s)}}(u - m + h^{(s)})$. Therefore $b_m(u), c_m(u)$ and $d_m(u)$ are polynomial in u . Now observe that

$$B_m(u), C_m(u) \in Y(\mathfrak{gl}_N)[[u^{-1}]] \cdot u^{-1}$$

by the definition (1.9). Since the degree of the polynomial $\rho_m(u)$ equals r_m the degrees of the polynomials $b_m(u)$ and $c_m(u)$ do not exceed $r_m - 1$ \square

In particular, $a_m(u)$ and $b_m(u), c_m(u), d_m(u)$ can be evaluated at any point $u \in \mathbb{C}$. Let us now fix for every $s = 1, \dots, n$ a scheme

$$\Lambda^{(s)} = (\lambda_{li}^{(s)} \mid 1 \leq i \leq l \leq M^{(s)} + N) \in \mathcal{T}_{\lambda^{(s)}, \mu^{(s)}}.$$

For every $m = 0, \dots, N$ consider the polynomial

$$\begin{aligned} \varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(n)}}(u) &= \rho_m(u) \cdot \chi_{m, \Lambda^{(1)}, \dots, \Lambda^{(n)}}(u) \\ &= \prod_{s=1}^n \prod_{i=1}^{M^{(s)}+m} (u + \lambda_{M^{(s)}+m, i}^{(s)} - i + 1 + h^{(s)}); \end{aligned}$$

see the definition (2.10). Note that all the r_m zeroes of this polynomial

$$\nu_{mi}^{(s)} = i - \lambda_{M^{(s)}+m, i}^{(s)} - 1 - h^{(s)}$$

are pairwise distinct due to our assumption on the parameters $h^{(1)}, \dots, h^{(n)}$. Also that we have the equality $\varpi_{0, \Lambda^{(1)}, \dots, \Lambda^{(n)}}(u) = \rho_0(u)$.

Endow the set of the pairs (l, j) where $l = 1, \dots, N$ and $j = 1, 2, \dots$ with the following relation of precedence: $(m, i) \prec (l, j)$ if $i < j$ or $i = j$ and $m > l$. Let ξ_0 be the singular vector in W constructed in the end of Section 2. Consider the vector in W

$$(3.3) \quad \xi = \prod_{(l, j)}^{\rightarrow} \left(\prod_{(r, t)} b_l(\nu_{lj}^{(r)} - t) \right) \cdot \xi_0$$

where the product in brackets is taken over the set \mathcal{F} of all pairs (r, t) such that

$$r = 1, \dots, n \quad \text{and} \quad j \leq M^{(r)} + l; \quad t = 1, \dots, \kappa_{M^{(r)}+l, j}^{(r)} - \lambda_{M^{(r)}+l, j}^{(r)}.$$

Here for each fixed l the elements $b_l(\nu_{lj}^{(r)} - t) \in \text{End}(W)$ commute because of the relation (1.18). The products in brackets do not commute with each other in general. We arrange them from the left to the right according to the above relation of precedence for the pairs (l, j) .

Theorem 3.2. *For every $m = 1, \dots, N$ we have the equality*

$$a_m(u) \cdot \xi = \varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(n)}}(u) \cdot \xi.$$

Proof. We will employ the induction on the number of the factors $b_l(\nu_{lj}^{(r)} - t)$ in (3.3). If there is no factors then $\Lambda^{(r)} = \Lambda_0^{(r)}$ for every $r = 1, \dots, n$ and $\xi = \xi_0$. In particular, the required equality then holds by Proposition 2.12.

Assume that the product in (3.3) contains at least one factor. Let (l, j) be the minimal pair such that the corresponding set \mathcal{F} is not empty. Let us fix an arbitrary $r \in \{1, \dots, n\}$ such that

$$\lambda_{M^{(r)}+l, j}^{(r)} < \kappa_{M^{(r)}+l, j}^{(r)}.$$

Denote by $\Omega^{(r)}$ the array obtained from $\Lambda^{(r)}$ by increasing the $(M^{(r)} + l, j)$ -entry by 1. Then $\Omega^{(r)} \in \mathcal{T}_{\lambda^{(r)}, \mu^{(r)}}$ and

$$(3.4) \quad \xi = b_l(\nu_{lj}^{(r)} - 1) \cdot \eta$$

where the vector η is determined in the way analogous to (3.3) by the sequence of schemes $\Lambda^{(1)}, \dots, \Omega^{(r)}, \dots, \Lambda^{(n)}$ instead of $\Lambda^{(1)}, \dots, \Lambda^{(r)}, \dots, \Lambda^{(n)}$. If $l \neq m$ then

$$\varpi_{m, \Lambda^{(1)}, \dots, \Omega^{(r)}, \dots, \Lambda^{(n)}}(u) = \varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(r)}, \dots, \Lambda^{(n)}}(u).$$

By the relation (1.16) and by the inductive assumption we get

$$\begin{aligned} a_m(u) \cdot \xi &= a_m(u) b_l(\nu_{lj}^{(r)} - 1) \cdot \eta = b_l(\nu_{lj}^{(r)} - 1) a_m(u) \cdot \eta = \\ &= b_l(\nu_{lj}^{(r)} - 1) \varpi_{m, \Lambda^{(1)}, \dots, \Omega^{(r)}, \dots, \Lambda^{(n)}}(u) \cdot \eta = \varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(r)}, \dots, \Lambda^{(n)}}(u) \cdot \xi. \end{aligned}$$

Now suppose that $l = m$; then by the definition of $\Omega^{(r)}$ we have

$$\varpi_{m, \Lambda^{(1)}, \dots, \Omega^{(r)}, \dots, \Lambda^{(n)}}(u) = \frac{u - \nu_{mj}^{(r)} + 1}{u - \nu_{mj}^{(r)}} \varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(r)}, \dots, \Lambda^{(n)}}(u).$$

In particular, by the inductive assumption we then have

$$a_m(\nu_{mj}^{(r)} - 1) \cdot \eta = \varpi_{m, \Lambda^{(1)}, \dots, \Omega^{(r)}, \dots, \Lambda^{(n)}}(\nu_{mj}^{(r)} - 1) \cdot \eta = 0.$$

Therefore by the relation (1.19) and again by the inductive assumption we get

$$\begin{aligned} a_m(u) \cdot \xi &= a_m(u) b_m(\nu_{mj}^{(r)} - 1) \cdot \eta \\ &= \frac{u - \nu_{mj}^{(r)}}{u - \nu_{mj}^{(r)} + 1} b_m(\nu_{mj}^{(r)} - 1) a_m(u) \cdot \eta \\ &= \frac{u - \nu_{mj}^{(r)}}{u - \nu_{mj}^{(r)} + 1} b_m(\nu_{mj}^{(r)} - 1) \varpi_{m, \Lambda^{(1)}, \dots, \Omega^{(r)}, \dots, \Lambda^{(n)}}(u) \cdot \eta \\ &= \varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(r)}, \dots, \Lambda^{(n)}}(u) \cdot \xi \quad \square \end{aligned}$$

Therefore the vector ξ determined by (3.3) satisfies the equality (2.11) for every $m = 1, \dots, N$. By Theorem 2.9 we can choose $\xi_{\Lambda^{(1)}, \dots, \Lambda^{(n)}} = \xi$ as soon as we prove that $\xi \neq 0$. We will describe first the action on the vector ξ of the coefficients of the polynomials $b_m(u)$ and $c_m(u)$ for $m = 1, \dots, N - 1$. Then we will prove that the vector ξ does not vanish.

Proposition 3.3. *If $\Lambda^{(r)} \notin \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ for some $r \in \{1, \dots, n\}$ then $\xi = 0$.*

Proof. As well as in the proof of Theorem 3.2 we will employ the induction on the number of the factors $b_l(\nu_{lj}^{(r)} - t)$ in (3.3). If there is no factors then $\Lambda^{(r)} = \Lambda_0^{(r)}$ for every $r = 1, \dots, n$ and we have nothing to prove.

Assume that the product in (3.3) contains at least one factor. Let (l, j) be the minimal pair such that the corresponding set \mathcal{F} is not empty. Fix an arbitrary $r \in \{1, \dots, n\}$ such that

$$\lambda_{M^{(r)}+l, j}^{(r)} < \kappa_{M^{(r)}+l, j}^{(r)}.$$

Introduce the array $\Omega^{(r)} \in \mathcal{T}_{\lambda^{(r)}, \mu^{(r)}}$ and determine the vector $\eta \in W$ in the same way as it was done in the proof of Theorem 3.2. Then we have the equality (3.4). Suppose that $\Omega^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ and $\Lambda^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ for any $s \neq r$. Otherwise $\eta = 0$ by the inductive assumption so that $\xi = 0$ due to (3.4).

By Theorem 3.2 the vector ξ is an eigenvector for the coefficients of the polynomials $a_1(u), \dots, a_{n-1}(u)$. Therefore by Theorem 2.9

$$\varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(r)}, \dots, \Lambda^{(n)}}(u) = \varpi_{m, \Upsilon^{(1)}, \dots, \Upsilon^{(r)}, \dots, \Upsilon^{(n)}}(u); \quad m = 1, \dots, N$$

for certain schemes

$$\Upsilon^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}; \quad r = 1, \dots, n.$$

Consider the roots $\nu_{mi}^{(r)}$ of the polynomial $\varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(r)}, \dots, \Lambda^{(n)}}(u)$. Since the array $\Omega^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ we have the inequalities

$$\begin{aligned} \nu_{m1}^{(r)} + h^{(r)} &< \nu_{m2}^{(r)} + h^{(r)} < \dots < \nu_{m, M^{(r)}+m}^{(r)} + h^{(r)} & \text{if } m \neq l; \\ \nu_{l1}^{(r)} + h^{(r)} &< \dots < \nu_{lj}^{(r)} + h^{(r)} \leq \nu_{l, j+1}^{(r)} + h^{(r)} < \dots < \nu_{l, M^{(r)}+l}^{(r)} + h^{(r)}. \end{aligned}$$

Moreover, we have $h^{(s)} - h^{(r)} \notin \mathbb{Z}$ for any $s \neq r$ by our assumption. Therefore the array $\Lambda^{(r)}$ can be uniquely restored from the collection of the polynomials

$$\varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(r)}, \dots, \Lambda^{(n)}}(u); \quad m = 1, \dots, N.$$

Thus $\Lambda^{(r)} = \Upsilon^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ and the Proposition 3.3 is proved \square

From now on we will suppose that $\Lambda^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ for each $s = 1, \dots, n$. Let an index $m \in \{1, \dots, N-1\}$ be fixed. By Proposition 3.1 to determine the polynomials $b_m(u) \cdot \xi$ and $c_m(u) \cdot \xi$ it suffices to evaluate them at r_m points. We will evaluate these polynomials at $u = \nu_{mi}^{(s)}$ where $s = 1, \dots, n$ and $i = 1, \dots, M^{(s)} + m$.

Let the indices s and i be fixed. Denote by $\Lambda_+^{(s)}$ the array obtained from $\Lambda^{(s)}$ by increasing the $(M^{(s)} + m, i)$ -entry by 1. If $\Lambda_+^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ then denote by ξ_+ the vector in W determined in the way analogous to (3.3) by the sequence of schemes $\Lambda^{(1)}, \dots, \Lambda_+^{(s)}, \dots, \Lambda^{(n)}$ instead of $\Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}$. Denote by $\gamma_{mi}^{(s)}$ the product

$$\prod_{r=1}^n \prod_{j=1}^{M^{(r)}+m+1} \begin{cases} (\kappa_{M^{(r)}+m+1, j}^{(r)} - \lambda_{M^{(s)}+m, i}^{(s)} + i - j + h^{(r)} - h^{(s)}) & \text{if } j \leq i \\ (\lambda_{M^{(r)}+m+1, j}^{(r)} - \lambda_{M^{(s)}+m, i}^{(s)} + i - j + h^{(r)} - h^{(s)}) & \text{if } j > i, \end{cases}$$

multiplied by the product

$$\prod_{r=1}^n \prod_{j=1}^{M^{(r)}+m-1} \begin{cases} (\kappa_{M^{(r)}+m-1,j}^{(r)} - \lambda_{M^{(s)}+m,i}^{(s)} + i - j - 1 + h^{(r)} - h^{(s)}) & \text{if } j < i \\ (\lambda_{M^{(r)}+m-1,j}^{(r)} - \lambda_{M^{(s)}+m,i}^{(s)} + i - j - 1 + h^{(r)} - h^{(s)}) & \text{if } j \geq i. \end{cases}$$

Lemma 3.4. *If $\Lambda_+^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ then $\gamma_{mi}^{(s)} \neq 0$.*

Proof. Any factor in the product $\gamma_{mi}^{(s)}$ corresponding to $r \neq s$ is not zero since $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ then. Since $\Lambda^{(s)}, \Lambda_0^{(s)}, \Lambda_+^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ we have the inequalities

$$\begin{aligned} \kappa_{M^{(s)}+m+1,j}^{(s)} &\geq \kappa_{M^{(s)}+m,j}^{(s)} \geq \kappa_{M^{(s)}+m,i}^{(s)} > \lambda_{M^{(s)}+m,i}^{(s)} && \text{if } j \leq i, \\ \lambda_{M^{(s)}+m+1,j}^{(s)} &\leq \lambda_{M^{(s)}+m,j-1}^{(s)} \leq \lambda_{M^{(s)}+m,i}^{(s)} && \text{if } j > i, \\ \kappa_{M^{(s)}+m-1,j}^{(s)} &\geq \kappa_{M^{(s)}+m,j+1}^{(s)} \geq \kappa_{M^{(s)}+m,i}^{(s)} > \lambda_{M^{(s)}+m,i}^{(s)} && \text{if } j < i, \\ \lambda_{M^{(s)}+m-1,j}^{(s)} &\leq \lambda_{M^{(s)}+m,j}^{(s)} \leq \lambda_{M^{(s)}+m,i}^{(s)} && \text{if } j \geq i. \end{aligned}$$

These inequalities show that any factor in the product $\gamma_{mi}^{(s)}$ corresponding to $r = s$ is also non-zero \square

Theorem 3.5. *We have*

$$c_m(\nu_{mi}^{(s)}) \cdot \xi = \begin{cases} -\gamma_{mi}^{(s)} \cdot \xi_+ & \text{if } \Lambda_+^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\Lambda^{(r)} = \Lambda_0^{(r)}$ for every $r = 1, \dots, n$ then $\Lambda_+^{(s)} \notin \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ in particular. On the other hand then $\xi = \xi_0$ and by Proposition 2.12 we have $c_m(\nu_{mi}^{(s)}) \cdot \xi = 0$.

Assume that the product in (3.3) contains at least one factor. Let (l, j) be the minimal pair such that the corresponding set \mathcal{F} is not empty. Fix an arbitrary $r \in \{1, \dots, n\}$ such that

$$\lambda_{M^{(r)}+l,j}^{(r)} < \kappa_{M^{(r)}+l,j}^{(r)}.$$

Introduce the array $\Omega^{(r)} \in \mathcal{T}_{\lambda^{(r)}, \mu^{(r)}}$ and determine the vector $\eta \in W$ in the same way as it was done in the proof of Theorem 3.2. Then we have the equality (3.4). Moreover, then $\Omega^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ since $\Lambda^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ by our assumption.

Let us prove first that the following equality holds for any $(l, j, r) \neq (m, i, s)$:

$$(3.5) \quad c_m(\nu_{mi}^{(s)}) b_l(\nu_{lj}^{(r)} - 1) \cdot \eta = b_l(\nu_{lj}^{(r)} - 1) c_m(\nu_{mi}^{(s)}) \cdot \eta.$$

If $l \neq m$ then we obtain (3.5) directly from the relation (1.17). But if $l = m$ then $\nu_{mj}^{(r)} - 1 \neq \nu_{mi}^{(s)}$. Indeed, for $r \neq s$ it follows from the assumption $h^{(r)} - h^{(s)} \notin \mathbb{Z}$. Since $\Omega^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ we also have $\nu_{mj}^{(r)} - 1 \neq \nu_{mi}^{(r)}$. Due to Theorem 3.2 we then also have the equalities

$$\begin{aligned} a_m(\nu_{mj}^{(r)} - 1) \cdot \eta &= \varpi_{m, \Lambda^{(1)}, \dots, \Omega^{(r)}, \dots, \Lambda^{(n)}}(\nu_{mj}^{(r)} - 1) \cdot \eta = 0, \\ a_m(\nu_{mi}^{(s)}) \cdot \eta &= \varpi_{m, \Lambda^{(1)}, \dots, \Omega^{(r)}, \dots, \Lambda^{(n)}}(\nu_{mi}^{(s)}) \cdot \eta = 0. \end{aligned}$$

Therefore by the relation (1.20) we again obtain that

$$c_m(\nu_{mi}^{(s)}) b_m(\nu_{mj}^{(r)} - 1) \cdot \eta = b_m(\nu_{mj}^{(r)} - 1) c_m(\nu_{mi}^{(s)}) \cdot \eta.$$

If $\lambda_{M^{(s)}+m,i}^{(s)} = \kappa_{M^{(s)}+m,i}^{(s)}$ then $\Lambda_+^{(s)} \notin \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$. On the other hand, by applying the equality (3.5) repeatedly we then get

$$\begin{aligned} c_m(\nu_{mi}^{(s)}) \cdot \xi &= c_m(\nu_{mi}^{(s)}) \prod_{(l,j)}^{\rightarrow} \left(\prod_{(r,t)} b_l(\nu_{lj}^{(r)} - t) \right) \cdot \xi_0 \\ &= \prod_{(l,j)}^{\rightarrow} \left(\prod_{(r,t)} b_l(\nu_{lj}^{(r)} - t) \right) c_m(\nu_{mi}^{(s)}) \cdot \xi_0 = 0 \end{aligned}$$

as we have claimed. Now we will assume that $\lambda_{M^{(s)}+m,i}^{(s)} < \kappa_{M^{(s)}+m,i}^{(s)}$.

For every $r = 1, \dots, n$ consider the array $\Upsilon^{(r)}$ the array obtained from $\Lambda^{(r)}$ by changing all the $(M^{(r)} + l, j)$ -entries corresponding to the pairs $(l, j) \prec (m, i)$ for $\kappa_{M^{(r)}+l,j}^{(r)}$ and also by increasing the (m, i) -entry by 1 if $r = s$. Then $\Upsilon^{(r)} \in \mathcal{T}_{\lambda^{(r)}, \mu^{(r)}}$ for any r . Determine the vector ζ in the way analogous to (3.3) by the sequence of schemes $\Upsilon^{(1)}, \dots, \Upsilon^{(n)}$ instead of $\Lambda^{(1)}, \dots, \Lambda^{(n)}$. Due to Theorem 3.2 we then have

$$a_m(\nu_{mi}^{(s)} - 1) \cdot \zeta = \varpi_{m, \Upsilon^{(1)}, \dots, \Upsilon^{(n)}}(\nu_{mi}^{(s)} - 1) \cdot \zeta = 0.$$

We then also have $\xi = p b_m(\nu_{mi}^{(s)} - 1) \cdot \zeta$ where

$$p = \prod_{(l,j) \prec (m,i)}^{\rightarrow} \left(\prod_{(r,t)} b_l(\nu_{lj}^{(r)} - t) \right).$$

Therefore by applying the equality (3.5) repeatedly and using Proposition 1.10 along with the equality (3.1) we get

$$\begin{aligned} c_m(\nu_{mi}^{(s)}) \cdot \xi &= p c_m(\nu_{mi}^{(s)}) b_m(\nu_{mi}^{(s)} - 1) \cdot \zeta = -p a_{m+1}(\nu_{mi}^{(s)}) a_{m-1}(\nu_{mi}^{(s)} - 1) \cdot \zeta = \\ &= -\varpi_{m+1, \Upsilon^{(1)}, \dots, \Upsilon^{(n)}}(\nu_{mi}^{(s)}) \varpi_{m-1, \Upsilon^{(1)}, \dots, \Upsilon^{(n)}}(\nu_{mi}^{(s)} - 1) p \cdot \zeta = -\gamma_{mi}^{(s)} p \cdot \zeta. \end{aligned}$$

Here if $\Lambda_+^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ then $p \cdot \zeta = \xi_+$ by definition. If $\Lambda_+^{(s)} \notin \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ then $p \cdot \zeta = 0$ by Proposition 3.3 \square

Proposition 3.6. *If $\Lambda^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ for every $r = 1, \dots, n$ then $\xi \neq 0$.*

Proof. As well as in the proof of Theorem 3.2 we will employ the induction on the number of the factors $b_l(\nu_{lj}^{(r)} - t)$ in (3.3). If there is no factors then $\Lambda^{(r)} = \Lambda_0^{(r)}$ for every $r = 1, \dots, n$ and $\xi = \xi_0 \neq 0$.

Assume that the product in (3.3) contains at least one factor. Let (l, j) be the minimal pair such that the corresponding set \mathcal{F} is not empty. Fix an arbitrary $r \in \{1, \dots, n\}$ such that

$$\lambda_{M^{(r)}+l,j}^{(r)} < \kappa_{M^{(r)}+l,j}^{(r)}.$$

Introduce the array $\Omega^{(r)} \in \mathcal{T}_{\lambda^{(r)}, \mu^{(r)}}$ and determine the vector $\eta \in W$ in the same way as it was done in the proof of Theorem 3.2. Then we have the equality (3.4). Since $\Lambda^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$ we also have $\Omega^{(r)} \in \mathcal{S}_{\lambda^{(r)}, \mu^{(r)}}$. Therefore $\eta \neq 0$ by the inductive assumption. On the other hand, by Theorem 3.5 we then have

$$c_l(\nu_{lj}^{(r)}) \cdot \xi = \gamma_{lj}^{(r)} \cdot \eta$$

where $\gamma_{lj}^{(r)} \neq 0$ due to Lemma 3.4. Therefore $\xi \neq 0$ \square

Now consider the array $\Lambda_-^{(s)}$ obtained from $\Lambda^{(s)}$ by decreasing the $(M^{(s)} + m, i)$ -entry by 1. If $\Lambda_-^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ then denote by ξ_- the vector in W determined in the way analogous to (3.3) by the sequence of schemes $\Lambda^{(1)}, \dots, \Lambda_-^{(s)}, \dots, \Lambda^{(n)}$ instead of $\Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}$. Denote by $\beta_{mi}^{(s)}$ the product

$$\begin{aligned} & \prod_{r=1}^n \prod_{j=1}^{\min(i, M^{(r)} + m + 1)} \frac{\lambda_{M^{(r)} + m + 1, j}^{(r)} - \lambda_{M^{(s)} + m, i}^{(s)} + i - j + 1 + h^{(r)} - h^{(s)}}{\kappa_{M^{(r)} + m + 1, j}^{(r)} - \lambda_{M^{(s)} + m, i}^{(s)} + i - j + 1 + h^{(r)} - h^{(s)}} \\ & \times \prod_{r=1}^n \prod_{j=1}^{\min(i, M^{(r)} + m) - 1} \frac{\lambda_{M^{(r)} + m - 1, j}^{(r)} - \lambda_{M^{(s)} + m, i}^{(s)} + i - j + h^{(r)} - h^{(s)}}{\kappa_{M^{(r)} + m - 1, j}^{(r)} - \lambda_{M^{(s)} + m, i}^{(s)} + i - j + h^{(r)} - h^{(s)}}. \end{aligned}$$

Lemma 3.7. *The product $\beta_{mi}^{(s)} \neq 0$ and is well defined.*

Proof. Any factor in the product $\beta_{mi}^{(s)}$ corresponding to $r \neq s$ is non-zero and well defined since $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ then. Since $\Lambda^{(s)}, \Lambda_0^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ we have the inequalities

$$\begin{aligned} \kappa_{M^{(s)} + m + 1, j}^{(s)} &\geq \lambda_{M^{(s)} + m + 1, j}^{(s)} \geq \lambda_{M^{(s)} + m, j}^{(s)} \geq \lambda_{M^{(s)} + m, i}^{(s)} & \text{if } j \leq i, \\ \kappa_{M^{(s)} + m - 1, j}^{(s)} &\geq \lambda_{M^{(s)} + m - 1, j}^{(s)} \geq \lambda_{M^{(s)} + m, j + 1}^{(s)} \geq \lambda_{M^{(s)} + m, i}^{(s)} & \text{if } j < i. \end{aligned}$$

These inequalities show that any factor in the product $\beta_{mi}^{(s)}$ corresponding to $r = s$ is also non-zero and well defined \square

Theorem 3.8. *We have*

$$b_m(\nu_{mi}^{(s)}) \cdot \xi = \begin{cases} \beta_{mi}^{(s)} \cdot \xi_- & \text{if } \Lambda_-^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will use again some of the arguments which appeared in the proofs of Theorem 3.2 and Proposition 3.3. Consider the vector $b_m(\nu_{mi}^{(s)}) \cdot \xi \in W$. Let us check first that for any $l = 1, \dots, N$ we have

$$(3.6) \quad a_l(u) b_m(\nu_{mi}^{(s)}) \cdot \xi = \varpi_{l, \Lambda^{(1)}, \dots, \Lambda_-^{(s)}, \dots, \Lambda^{(n)}}(u) b_m(\nu_{mi}^{(s)}) \cdot \xi.$$

Indeed, if $l \neq m$ then

$$\varpi_{l, \Lambda^{(1)}, \dots, \Lambda_-^{(s)}, \dots, \Lambda^{(n)}}(u) = \varpi_{l, \Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}}(u).$$

On the other hand, by the relation (1.16) and by Theorem 3.2 we then have

$$a_l(u) b_m(\nu_{mi}^{(s)}) \cdot \xi = b_m(\nu_{mi}^{(s)}) a_l(u) \cdot \xi = \varpi_{l, \Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}}(u) b_m(\nu_{mi}^{(s)}) \cdot \xi.$$

Suppose that $l = m$; then by the definition of $\Lambda_-^{(s)}$ we have

$$\varpi_{m, \Lambda^{(1)}, \dots, \Lambda_-^{(s)}, \dots, \Lambda^{(n)}}(u) = \frac{u - \nu_{mj}^{(s)} - 1}{u - \nu_{mj}^{(s)}} \varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}}(u).$$

Since due to Theorem 3.2

$$(3.7) \quad a_m(\nu_{mi}^{(s)}) \cdot \xi = \varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}}(\nu_{mi}^{(s)}) \cdot \xi = 0,$$

by the relation (1.19) and again by Theorem 3.2 we get the equalities

$$\begin{aligned} a_m(u) b_m(\nu_{mi}^{(s)}) \cdot \xi &= \frac{u - \nu_{mi}^{(s)} - 1}{u - \nu_{mi}^{(s)}} b_m(\nu_{mi}^{(s)}) a_m(u) \cdot \xi \\ &= \varpi_{m, \Lambda^{(1)}, \dots, \Lambda_-^{(s)}, \dots, \Lambda^{(n)}}(u) b_m(\nu_{mi}^{(s)}) \cdot \xi. \end{aligned}$$

Since $\Lambda^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ the array $\Lambda_-^{(s)}$ can be uniquely restored from the collection of the polynomials

$$\varpi_{l, \Lambda^{(1)}, \dots, \Lambda_-^{(s)}, \dots, \Lambda^{(n)}}(u); \quad l = 1, \dots, N.$$

Thus if $\Lambda_-^{(s)} \notin \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$ then $b_m(\nu_{mi}^{(s)}) \cdot \xi = 0$ by Theorem 2.9 as we have claimed.

Assume that $\Lambda_-^{(s)} \in \mathcal{S}_{\lambda^{(s)}, \mu^{(s)}}$. Then due to Theorem 2.9 and Proposition 3.6 the equalities (3.6) imply that

$$(3.8) \quad b_m(\nu_{mi}^{(s)}) \cdot \xi = \beta \cdot \xi_-$$

for some $\beta \in \mathbb{C}$. We will prove that $\beta = \beta_{mi}^{(s)}$ here.

Let us compare the action of the element $c_m(\nu_{mi}^{(s)} + 1)$ on the both sides of the equality (3.8). On the right hand side by Theorem 3.5 we have

$$c_m(\nu_{mi}^{(s)} + 1) \beta \cdot \xi = -\beta \gamma \cdot \xi$$

where γ is analogue of $\gamma_{mi}^{(s)}$ for the collection of schemes $\Lambda^{(1)}, \dots, \Lambda_-^{(s)}, \dots, \Lambda^{(n)}$ instead of $\Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}$. In particular, here $\gamma \neq 0$ by Lemma 3.4. On the left hand side by applying Proposition 1.10 and using the equality (3.7) we get

$$\begin{aligned} c_m(\nu_{mi}^{(s)} + 1) b_m(\nu_{mi}^{(s)}) \cdot \xi &= -a_{m+1}(\nu_{mi}^{(s)} + 1) a_{m-1}(\nu_{mi}^{(s)}) \cdot \xi \\ &= -\varpi_{m+1, \Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}}(\nu_{mi}^{(s)} + 1) \varpi_{m-1, \Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}}(\nu_{mi}^{(s)}) \cdot \xi. \end{aligned}$$

Since $\xi \neq 0$ by Proposition 3.6, we finally obtain that

$$\beta = \varpi_{m+1, \Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}}(\nu_{mi}^{(s)} + 1) \varpi_{m-1, \Lambda^{(1)}, \dots, \Lambda^{(s)}, \dots, \Lambda^{(n)}}(\nu_{mi}^{(s)}) \gamma^{-1} = \beta_{mi}^{(s)}.$$

Thus we have proved Theorem 3.8 \square

Due to Proposition 3.6 we can choose $\xi_{\Lambda^{(1)}, \dots, \Lambda^{(n)}} = \xi$. Theorems 3.2 and 3.5, 3.8 then completely describe the action of the Drinfeld generators of the algebra $Y(\mathfrak{gl}_N)$ in the module W . Since the vector $\xi_{\Lambda^{(1)}, \dots, \Lambda^{(n)}}$ is uniquely determined by the equalities (2.11), we obtain the following corollary to Lemmas 3.4, 3.7.

Corollary 3.9. *$Y(\mathfrak{gl}_N)$ -module W is irreducible if $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$.*

4. Classification theorem

Let V be any irreducible finite-dimensional module over the algebra $Y(\mathfrak{gl}_N)$. Let $P_1(u), \dots, P_{N-1}(u)$ be the Drinfeld polynomials corresponding to V . For each $m = 1, \dots, N-1$ consider the collection of zeroes of the polynomial $P_m(-u)$

$$(z_{mi} \mid i = 1, \dots, \deg P_m).$$

Theorem 4.1. *The following three conditions are equivalent.*

- a) *For all $m \geq l$ we have $z_{li} - z_{mj} \neq 0, \dots, m-l$ unless $(m, i) = (l, j)$.*
- b) *Up to some automorphism ω_f of $Y(\mathfrak{gl}_N)$ the module V has the form (2.9) where $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$.*
- c) *The action in the module V of the subalgebra $A(\mathfrak{gl}_N)$ is semisimple.*

Proof. Due to Theorem 2.9 the condition (b) implies (c). We will demonstrate that (a) implies (b) and that (c) implies (a). Suppose that the condition (a) is satisfied. In particular, then we have $u_{mi} \neq u_{lj}$ unless $(m, i) = (l, j)$. Consider the partition of the set of all zeroes

$$\mathcal{Z} = \{z_{mi} \mid m = 1, \dots, N-1; i = 1, \dots, \deg P_m\} = \mathcal{Z}^{(1)} \sqcup \dots \sqcup \mathcal{Z}^{(n)}$$

where $\mathcal{Z}^{(1)}, \dots, \mathcal{Z}^{(n)}$ are the maximal subsets in \mathcal{Z} such that

$$z, w \in \mathcal{Z}^{(s)} \Rightarrow z - w \in \mathbb{Z}; \quad s = 1, \dots, n.$$

Furthermore for each $s = 1, \dots, n$ introduce the partition

$$\mathcal{Z}^{(s)} = \mathcal{Z}_1^{(s)} \sqcup \dots \sqcup \mathcal{Z}_{N-1}^{(s)}$$

into the subsets of zeroes of the polynomials $P_1(-u), \dots, P_{N-1}(-u)$ respectively. We will suppose that $\mathcal{Z} \neq \emptyset$; otherwise the module V is one-dimensional and there we have $T_{ij}(u) \mapsto f(u) \cdot \delta_{ij}$ for certain series $f(u) \in 1 + u^{-1} \mathbb{C}[[u^{-1}]]$. For each $s = 1, \dots, n$ we will point out a parameter $h^{(s)} \in \mathbb{C}$ and a skew Young diagram $\lambda^{(s)}/\mu^{(s)}$ whose columns are of the height at most $N-1$ such that

$$(4.1) \quad P_{m, \lambda^{(s)}/\mu^{(s)}}(u + h^{(s)}) = \prod_{z \in \mathcal{Z}_m^{(s)}} (u + z); \quad m = 1, \dots, N-1.$$

Then by Corollary 2.13 the module V will have the same Drinfeld polynomials as the module (2.9). Therefore the condition (b) will be also satisfied.

Let any $s \in \{1, \dots, n\}$ be fixed such that $\mathcal{Z}^{(s)} \neq \emptyset$. Denote $\# \mathcal{Z}^{(s)} = p$. Let z_1, \dots, z_p be the elements of the set $\mathcal{Z}^{(s)}$; we will assume that $z_k - z_{k-1} < 0$ for all possible indices k . Put $h^{(s)} = z_1$. Suppose that $z_1 \in \mathcal{Z}_q^{(s)}$. Then assign to each $k = 1, \dots, p$ the column

$$(4.2) \quad \{ (i, j) \in \mathbb{Z}^2 \mid j = q - k + 1, \ z_k - z_1 \leq j - i < z_k - z_1 + m \}$$

where $z_k \in \mathcal{Z}_m^{(s)}$. Note that here for $k = 1$ we have $i \geq 1$. By our assumption for any $k > 1$ we have the inequality

$$(4.3) \quad z_k - z_1 < z_{k-1} - z_1.$$

Moreover, if $z_{k-1} \in \mathcal{Z}_l^{(s)}$ we also have the inequality

$$(4.4) \quad z_k - z_1 + m < z_{k-1} - z_1 + l.$$

If $m \leq l$ it follows from (4.3). If $m > l$ then $z_{k-1} - z_k \neq 1, \dots, m - l$ by the condition (a) so that (4.4) again follows from (4.3). Therefore the collection of all columns (4.2) is a skew Young diagram. Denote this diagram by $\lambda^{(s)}/\mu^{(s)}$; the equalities (4.1) then evidently hold. Thus we have proved that (a) implies (b).

It remains to prove that the condition (c) implies (a). We will consider first the case $N = 2$. Denote by $V_k(h)$ the $Y(\mathfrak{gl}_2)$ -module $V_{\lambda, \mu}(h)$ with $\lambda = (k, 0)$ and $\mu = \emptyset$. Then up to some automorphism ω_f every irreducible finite-dimensional $Y(\mathfrak{gl}_2)$ -module has the form

$$W = V_{k^{(1)}}(h^{(1)}) \otimes \dots \otimes V_{k^{(n)}}(h^{(n)})$$

for certain $k^{(1)}, \dots, k^{(n)} \in \{1, 2, \dots\}$ and $h^{(1)}, \dots, h^{(n)} \in \mathbb{C}$. Moreover, if we denote

$$\mathcal{X}^{(s)} = \{ h^{(s)}, h^{(s)} + 1, \dots, h^{(s)} + k^{(s)} \}; \quad s = 1, \dots, n$$

then the condition $\mathcal{X}^{(r)} \cap \mathcal{X}^{(s)} \neq \emptyset$ implies that either $\mathcal{X}^{(r)} \subset \mathcal{X}^{(s)}$ or $\mathcal{X}^{(r)} \supset \mathcal{X}^{(s)}$. These two facts are contained in [T2]; see also [CP1], Theorems 4.1 and 4.11. Furthermore, by Proposition 2.8 for any choice of the elements $x^{(s)} \in \mathcal{X}^{(s)}$ with $s = 1, \dots, n$ there exists a non-zero vector $\xi \in W$ such that

$$(4.5) \quad A_1(u) \cdot \xi = \xi \cdot \prod_{s=1}^n \frac{u + x^{(s)}}{u + h^{(s)}}, \quad A_2(u) \cdot \xi = \xi \cdot \prod_{s=1}^n \frac{u + k^{(s)} + h^{(s)}}{u + h^{(s)}}.$$

Put $\rho_1(u) = (u + h^{(1)}) \dots (u + h^{(n)})$. Consider the formal Laurent series in u^{-1}

$$(4.6) \quad \rho_1(u) A_1(u), \quad \rho_1(u) B_1(u), \quad \rho_1(u) C_1(u), \quad \rho_1(u) D_1(u).$$

By the definition of the comultiplication (1.23) the action in W of every coefficient of these series at u^{-1}, u^{-2}, \dots vanishes. Denote by $a(u), b(u), c(u), d(u)$ the polynomials in u with the coefficients in $\text{End}(W)$ corresponding to the series (4.6).

Suppose that the Drinfeld polynomial $P(u)$ of the $Y(\mathfrak{gl}_2)$ -module W has multiple zeroes. We have to show that the action in W of the subalgebra $A(\mathfrak{gl}_2)$ is then not semisimple. Due to Corollary 2.13 we have

$$P(u) = \prod_{s=1}^n \prod_{j=1}^{k^{(s)}} (u + j - 1 + h^{(s)}).$$

Let z be the zero of the polynomial $P(-u)$ of the maximal multiplicity. For some $r \in \{1, \dots, n\}$ we have $z \in \mathcal{X}^{(r)}$. We will assume that the set $\mathcal{X}^{(r)}$ is minimal amongst those of $\mathcal{X}^{(1)}, \dots, \mathcal{X}^{(n)}$ which contain z . Then the condition $\mathcal{X}^{(r)} \cap \mathcal{X}^{(s)} \neq \emptyset$ implies that $\mathcal{X}^{(r)} \subset \mathcal{X}^{(s)}$. In particular, for $x = k^{(r)} + h^{(r)}$ we have

$$(4.7) \quad x \neq h^{(s)}, k^{(s)} + h^{(s)} + 1; \quad s = 1, \dots, n.$$

Now set

$$x^{(s)} = \begin{cases} x & \text{if } x \in \mathcal{X}^{(s)}; \\ h^{(s)} & \text{otherwise} \end{cases}$$

and consider a non-zero vector $\xi \in W$ satisfying the corresponding equalities (4.5). Put $\alpha(u) = (u + x^{(1)}) \dots (u + x^{(n)})$. Then we have $a(u) \cdot \xi = \alpha(u) \cdot \xi$. In particular,

$$(4.8) \quad a(-x) \cdot \xi = \alpha(-x) \cdot \xi = 0, \quad \frac{da}{du}(-x) \cdot \xi = \frac{d\alpha}{du}(-x) \cdot \xi = 0.$$

Introduce the vectors in W

$$\xi' = b(-x) \cdot \xi, \quad \xi'' = \frac{db}{du}(-x) \cdot \xi.$$

We shall prove that

$$(4.9) \quad a(u) \cdot \xi' = \alpha(u) \frac{u + x - 1}{u + x} \cdot \xi';$$

$$(4.10) \quad a(u) \cdot \xi'' = \alpha(u) \frac{u + x - 1}{u + x} \cdot \xi'' - \frac{\alpha(u)}{(u + x)^2} \cdot \xi'.$$

and that $\xi' \neq 0$. Then the equalities (4.9), (4.10) will imply that the action in W of the subalgebra in $Y(\mathfrak{gl}_2)$ generated by the coefficients of $A_1(u)$ is not semisimple.

Due to (4.8) by the relation (1.19) we have

$$\begin{aligned} a(u) \cdot \xi' &= a(u) b(-x) \cdot \xi = \frac{u + x - 1}{u + x} b(-x) a(u) \cdot \xi \\ &= \alpha(u) \frac{u + x - 1}{u + x} b(-x) \cdot \xi = \alpha(u) \frac{u + x - 1}{u + x} \cdot \xi'; \end{aligned}$$

$$\begin{aligned} a(u) \cdot \xi'' &= a(u) \frac{db}{du}(-x) \cdot \xi \\ &= \frac{u + x - 1}{u + x} \frac{db}{du}(-x) a(u) \cdot \xi + \frac{1}{u + x} [a(u), b(-x)] \cdot \xi \\ &= \alpha(u) \frac{u + x - 1}{u + x} \frac{db}{du}(-x) \cdot \xi - \frac{\alpha(u)}{(u + x)^2} b(-x) \cdot \xi \\ &= \alpha(u) \frac{u + x - 1}{u + x} \cdot \xi'' - \frac{\alpha(u)}{(u + x)^2} \cdot \xi'. \end{aligned}$$

Thus we have proved (4.9) and (4.10). Furthermore, due to the second equality in (4.5) by the relation (1.21) we have

$$\begin{aligned} c(1 - x) \cdot \xi' &= c(1 - x) b(-x) \cdot \xi = \rho_1(1 - x) \rho_1(-x) A_2(1 - x) \cdot \xi \\ &= \xi \cdot \prod_{s=1}^n (k^{(s)} + h^{(s)} + 1 - x) (h^{(s)} - x). \end{aligned}$$

Every factor in the above product differs from zero due to (4.7) so that $\xi' \neq 0$.

Thus if the action of $A(\mathfrak{gl}_2)$ in an irreducible finite-dimensional $Y(\mathfrak{gl}_2)$ -module is semisimple then the corresponding Drinfeld polynomial has no multiple zeroes. Hence (c) implies (a) for $N = 2$. Let us now prove that (c) implies (a) for $N \geq 3$.

Let $\xi \in V$ be a singular vector. Suppose that the action in the module V of the subalgebra $A(\mathfrak{gl}_N)$ is semisimple. Note that due to (1.14) the action of $A(\mathfrak{gl}_N)$ in the $Y(\mathfrak{gl}_N)$ -module \bar{V} is then also semisimple.

Let the indices $m, l \in \{1, \dots, N-1\}$ such that $m \geq l$ be fixed. Consider the embedding $\varphi_{l,m+1}$ of the algebra $Y(\mathfrak{gl}_2)$ into $Y(\mathfrak{gl}_N)$. By Proposition 1.14 the action of $A(\mathfrak{gl}_2)$ in V corresponding to this embedding is semisimple. The action of $A(\mathfrak{gl}_2)$ in the irreducible $Y(\mathfrak{gl}_2)$ -subquotient of V generated by the vector ξ is then also semisimple. But due to Proposition 2.10 in the module V we have

$$\begin{aligned} & \frac{T_{m+1,m+1}(u-1)}{T_{l,l}(u-1)} \cdot \xi = \\ & \frac{T_{m+1,m+1}(u-1)}{T_{m,m}(u-1)} \cdot \frac{T_{m,m}(u-1)}{T_{m-1,m-1}(u-1)} \cdot \dots \cdot \frac{T_{l+1,l+1}(u-1)}{T_{l,l}(u-1)} \cdot \xi = \\ & \frac{P_m(u+m-2)}{P_m(u+m-1)} \cdot \frac{P_{m-1}(u+m-3)}{P_{m-1}(u+m-2)} \cdot \dots \cdot \frac{P_l(u+l-2)}{P_l(u+l-1)} \cdot \xi \end{aligned}$$

so that the Drinfeld polynomial of the above introduced subquotient equals

$$P_m(u+m-1) P_{m-1}(u+m-2) \dots P_l(u+l-1).$$

As we have already proved all zeroes of this polynomial are simple. Therefore

$$(4.11) \quad z_{li} - z_{mj} \neq m-l \quad \text{unless} \quad (m,i) = (l,j).$$

By Proposition 2.11 the Drinfeld polynomial of the irreducible $Y(\mathfrak{gl}_2)$ -subquotient of the module \bar{V} generated by the vector ξ then equals

$$\begin{aligned} & \bar{P}_m(u+m-1) \bar{P}_{m-1}(u+m-2) \dots \bar{P}_l(u+l-1) = \\ & P_{N-m}(u+N-1) P_{N-m+1}(u+N-1) \dots P_{N-l}(u+N-1). \end{aligned}$$

All zeroes of this polynomial are also simple. Replacing here $N-m$ and $N-l$ by l and m respectively we obtain that the polynomials $P_l(u)$ and $P_m(u)$ of the module V have no common zeroes.

Let us now assume that $m > l$. Fix an arbitrary $k \in \{1, \dots, m-l\}$ and put

$$\mathbf{k} = (l, l+1, \dots, l+k, m+1).$$

Consider the embedding $\varphi_{\mathbf{k}}$ of the algebra $Y(\mathfrak{gl}_{k+2})$ into $Y(\mathfrak{gl}_N)$. By Proposition 1.14 the action of $A(\mathfrak{gl}_{k+2})$ in V corresponding to this embedding is semisimple. The action of $A(\mathfrak{gl}_{k+2})$ in the irreducible $Y(\mathfrak{gl}_{k+2})$ -subquotient of V generated by the vector ξ is then also semisimple. Due to Proposition 2.10 the first and the last Drinfeld polynomials of this subquotient are respectively $P_l(u+l-1)$ and

$$P_m(u+m-k-1) P_{m-1}(u+m-k-2) \dots P_{l+k}(u+l-1).$$

As we have already proved, these two polynomials have no common zeroes. Thus

$$(4.12) \quad z_{li} - z_{mj} \neq m - l - k \quad \text{unless} \quad (m, i) = (l, j); \quad k = 1, \dots, m - l.$$

The statements (4.11) and (4.12) constitute the condition a). We have completed the proof of Theorem 4.1 \square

References

- [C] *I. Cherednik*, A new interpretation of Gelfand-Zetlin bases, *Duke Math. J.* **54** (1987), 563–577.
- [CP1] *V. Chari and A. Pressley*, Yangians and R -matrices, *L'Enseign. Math.* **36** (1990), 267–302.
- [CP2] *V. Chari and A. Pressley*, Fundamental representations of Yangians and singularities of R -matrices, *J. Reine Angew. Math.* **417** (1991), 87–128.
- [D1] *V. Drinfeld*, Hopf algebras and the quantum Yang–Baxter equation, *Soviet Math. Dokl.* **32** (1985), 254–258.
- [D2] *V. Drinfeld*, A new realization of Yangians and quantized affine algebras, *Soviet Math. Dokl.* **36** (1988), 212–216.
- [D3] *V. G. Drinfeld*, Quantum Groups, in “Proceedings of the International Congress of Mathematicians”, Amer. Math. Soc., Providence, 1987, 798–820.
- [GRV] *V. Ginzburg, N. Reshetikhin and E. Vasserot*, Quantum groups and flag varieties, *Contemp. Math.* **175** (1994), 101–130.
- [GZ] *I. Gelfand and M. Zetlin*, Finite-dimensional representations of the unimodular group, *Dokl. Akad. Nauk SSSR* **71** (1950), 825–828.
- [M] *A. Molev*, Gelfand-Tsetlin bases for representations of Yangians, *Lett. Math. Phys.* **30** (1994), 53–60.
- [MNO] *A. Molev, M. Nazarov and G. Olshanski*, Yangians and classical Lie algebras, *Russian Math. Surveys* **51** (1996), 205–282.
- [NT] *M. Nazarov and V. Tarasov*, Yangians and Gelfand-Zetlin bases, *Publ. RIMS* **30** (1994), 459–478.
- [O] *G. Olshanski*, Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians, *Adv. Soviet Math.* **2**, 1991, 1–66.
- [T1] *V. Tarasov*, Structure of quantum L -operators for the R -matrix of the XXZ -model, *Theor. Math. Phys.* **61** (1984), 1065–1071.
- [T2] *V. Tarasov*, Irreducible monodromy matrices for the R -matrix of the XXZ -model and lattice local quantum Hamiltonians, *Theor. Math. Phys.* **63** (1985), 440–454.

Mathematics Department, University of York, Heslington, York YO1 5DD, England
e-mail: mln1@york.ac.uk

Steklov Mathematical Institute, Fontanka 27, St. Petersburg 191011, Russia
e-mail: vt@pdmi.ras.ru